8th PhD Summer School in Discrete Mathematics
Vertex-transitive graphs and their local actions II

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Vertex-stabilisers

Lemma (Orbit-stabiliser)

If G is a transitive group of degree n, then $|G| = n|G_v|$.

Γ	$G = \mathrm{Aut}(\Gamma)$	G_v
C_n	D_n	C_2
K_n	$\mathrm{Sym}(n)$	$\mathrm{Sym}(n-1)$
$\mathrm{K}_{n,n}$	$\mathrm{Sym}(n) \wr \mathrm{Sym}(2)$	$\mathrm{Sym}(n-1) \times \mathrm{Sym}(n)$
$\mathrm{K}_{m[n]}$	$\mathrm{Sym}(n) \wr \mathrm{Sym}(m)$	$\mathrm{Sym}(n-1) \times (\mathrm{Sym}(n) \wr \mathrm{Sym}(m-1))$
$\mathrm{C}_n \square \mathrm{K}_2$	$\mathrm{D}_n \times \mathrm{C}_2$	C_2
$n \neq 4$		
Q_3	$\mathrm{C}_2 \wr \mathrm{Sym}(3)$	$\mathrm{Sym}(3)$
Pet	$\mathrm{Sym}(5)$	$\mathrm{Sym}(2) \times \mathrm{Sym}(3)$

Structure of vertex-stabilisers

Lemma

Let Γ be a connected graph of maximal valency k with an automorphism fixing a vertex and having order a prime p . Then $p \leq k$.

Proof.

Suppose, by contradiction, that $p > k$. Let g be an automorphism of order p fixing a vertex v. There is an induced action of g on $\Gamma(v)$. Since $|\Gamma(v)| \leq k < p$, g acts trivially on $\Gamma(v)$ and thus fixes all neighbours of v. Using connectedness and repeating this argument yields that g fixes all vertices of Γ, a contradiction. □

Tutte's Theorem and applications

Theorem (Tutte 1947)

If Γ is a connected 3-valent G-arc-transitive graph, then there exists $s \in \{1, \ldots, 5\}$ such that Γ is G-s-arc-regular.

s	1	2	3	4	5
G_v	C_3	$\mathrm{Sym}(3)$	$\mathrm{Sym}(3) \times \mathrm{Sym}(2)$	$\mathrm{Sym}(4)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(2)$
$\|G_v\|$	3	6	12	24	48

$|G_v| \leq 48$, so $|G| \leq 48|\mathrm{V}(\Gamma)|$.

Application of Tutte

Theorem (Potočnik,Spiga,V 2017)

The number of 3-valent arc-transitive graphs of order at most n is at most

$$n^{5+4b\log n} \sim n^{c\log n}.$$

Proof.

Let Γ be a 3-valent arc-transitive graph of order at most n and let $A = \mathrm{Aut}(\Gamma)$. Note that $|A| \leq 48n < n^2$ and A is 2-generated. By a result of Lubotzky, there exists b such that the number of isomorphism classes for A is at most $(n^2)^{b\log n^2} = n^{4b\log n}$. A_v is 2-generated, so at most $(n^2)^2 = n^4$ choices for A_v. At most n choices for a neighbour of v, and this determines Γ. □

There also exists c' such that the number is at least $n^{c'\log n}$. This also relies on Tutte's Theorem.

Application of Tutte II

Each pair (Γ, G) occurs as a finite quotient of an (infinite) group amalgam acting on the (infinite) cubic tree. By Tutte, there are only finitely many amalgams to consider, and the index is linear in the order of the graph.

This allows one (for example Conder) to enumerate these graphs up to "large" order (in this case, 10000).

`https://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt`

Application of Tutte III

Theorem (Conder, Li, Potočnik 2015)

Let k be a positive integer. There are only finitely many 3-valent 2-arc-transitive graphs of order kp with p a prime.

Proof.

Let $p > 48k$ be prime, Γ be a 3-valent 2-arc-transitive graph of order kp and $G = \mathrm{Aut}(\Gamma)$. Then $|G| = kp|G_v| \leq 48kp$. By Sylow, G has a normal Sylow p-subgroup P. Let C be the centraliser of P in G. By Schur-Zassenhaus, $C = P \times J$ for some J. Since $|P|$ and $|J|$ are coprime, J is characteristic in C and normal in G and

$$C_v = C \cap G_v = (P \times J) \cap G_v = (P \cap G_v) \times (J \cap G_v) = P_v \times J_v.$$

Since $P_v = 1$, we have $C_v = J_v$. Suppose $J_v \neq 1$. By Locally Quasiprimitive Lemma, J has at most two orbits of the same size, which is divisible by p since $p > 2$. This contradicts the fact that $|J|$ is coprime to p. It follows that $C_v = J_v = 1$, and thus G_v embeds into $\mathrm{Aut}(P)$ which is cyclic. Contradiction. □

Generalisation to 4-valent?

The wreath graph $\mathrm{W}_m = \mathrm{C}_m[\mathrm{K}_2^c]$ is the lexicographic product of a cycle of length m with an edgeless graph on 2 vertices.

We have $G = \mathrm{C}_2 \wr \mathrm{D}_m \leq \mathrm{Aut}(\mathrm{W}_m)$.

So W_m is a 4-valent arc-transitive graph, $|\mathrm{V}(\mathrm{W}_m)| = 2m$, $|G| = m2^{m+1}$, so $|G_v| = 2^m$.

$|G_v|$ is exponential in $|\mathrm{V}(\mathrm{W}_m)|$.

Generalisation to vertex-transitive?

The split wreath graph SW_m is a 3-valent vertex-transitive graph.

$|\mathrm{V}(SW_m)| = 4m$, $|G| = m2^{m+1}$, so $|G_v| = 2^{m-1}$.

Local action

Let Γ be a connected G-vertex-transitive graph.

Let $L = G_v^{\Gamma(v)}$, the permutation group induced by G_v on the neighbourhood $\Gamma(v)$.

We say that (Γ, G) is locally-L.

$G_v^{\Gamma(v)}$ is a permutation group of degree the valency of Γ and does not depend on v.

Let $G_v^{[1]}$ be the subgroup of G consisting of elements fixing v and all its neighbours.

$G_v^{\Gamma(v)} \cong G_v / G_v^{[1]}$.

Examples

Γ	$\mathrm{Aut}(\Gamma)_v$	$\mathrm{Aut}(\Gamma)_v^{\Gamma(v)}$
C_n	C_2	C_2
K_n	$\mathrm{Sym}(n-1)$	$\mathrm{Sym}(n-1)$
$\mathrm{K}_{n,n}$	$\mathrm{Sym}(n-1) \times \mathrm{Sym}(n)$	$\mathrm{Sym}(n)$
$\mathrm{C}_n \Box \mathrm{K}_2$	C_2	C_2
$n \neq 4$		
Q_3	$\mathrm{Sym}(3)$	$\mathrm{Sym}(3)$
Pet	$\mathrm{Sym}(2) \times \mathrm{Sym}(3)$	$\mathrm{Sym}(3)$

Some basic results

Lemma

If $H \leq G$, then $H_v \leq G_v$ and $H_v^{\Gamma(v)} \leq G_v^{\Gamma(v)}$.

If $N \trianglelefteq G$, then $N_v \trianglelefteq G_v$ and $N_v^{\Gamma(v)} \trianglelefteq G_v^{\Gamma(v)}$.

Theorem

Let (Γ, G) be a locally-L pair.

1. *L is transitive $\iff$ G is arc-transitive.*
2. *L is 2-transitive $\iff$ G is 2-arc-transitive.*

Proof.

Exercises. □

The Leash

Lemma

Let (Γ, G) *be a locally-L pair and* (u, v) *be an arc of* Γ*. There is a subnormal series for* G_v

$$1 = G_n \trianglelefteqslant G_{n-1} \trianglelefteqslant \cdots \trianglelefteqslant G_1 \trianglelefteqslant G_0 = G_v$$

such that $G_0/G_1 \cong L$ *and, for* $i \geq 1$, $G_i/G_{i+1} \preceq L_x$*.*

Also, $G_1 \trianglelefteqslant G_{(u,v)}$*, with* $G_{(u,v)}/G_1 \preceq L_x$*.*

Proof.

Let $(v = v_1, \ldots, v_n)$ be a walk including all vertices of Γ (possibly with repetition). Let $G_0 = G_{v_1}$ and for $i \geq 1$, let
$G_i = G_{v_1}^{[1]} \cap \cdots \cap G_{v_i}^{[1]}$. □

Corollaries

A permutation group G on X is semiregular if $G_x = 1$ for all $x \in X$. Equivalently, for $x, y \in X$, there is at most one $g \in G$ such that $x^g = y$. In this case, $|G|$ divides $|X|$.

Regular $\iff$ transitive $+$ semiregular.

Corollary

Let (Γ, G) be a locally-L pair and (u, v) be an arc of Γ.

1. *If the valency is a prime p, then $|G_{uv}|$ is not divisible by p and $|G_v|$ is not divisible by p^2.*
2. *L is semiregular $\iff$ G is arc-semiregular.*
3. *G_v is soluble $\iff$ L is soluble.*
4. *G_{uv} is soluble $\iff$ L_x is soluble.*

Quasiprimitive and semiprimitive groups

Definition

A permutation group is quasiprimitive if all its nontrivial normal subgroups are transitive. A group is semiprimitive if every normal subgroup is transitive or semiregular.

Lemma

Primitive $\Longrightarrow$ *Quasiprimitive* $\Longrightarrow$ *Semiprimitive*

Proof.

Exercise. □

Examples

1. Any transitive simple group is quasiprimitive. (For example, the group of rotation of the dodecahedron, acting on its faces, is QP but not P.)
2. Dihedral groups? (Exercise.)
3. $\mathrm{GL}(V)$ acting on a vector space V. (SP but not QP)

Back to bounding $|G_v|$

Theorem (Gardiner 1973)

Let Γ be 4-valent and (Γ, G) be locally-$\mathrm{Alt}(4)$ or $\mathrm{Sym}(4)$. Then $|G_v| \leq 2^4 \cdot 3^6$.

We can use this to prove results analogous to corollaries of Tutte.

Corollary

Let Γ be a 4-valent G-arc-transitive graph, and let L be the local action. The possibilities are:

L	L_x	$\|G_v\|$
C_4	1	4
C_2^2	1	4
D_4	C_2	2^x
$\mathrm{Alt}(4)$	C_3	$\leq 2^2 \cdot 3^4$
$\mathrm{Sym}(4)$	$\mathrm{Sym}(3)$	$\leq 2^4 \cdot 3^6$

The only "problem" is the locally-D_4 case. (As in W_m.)

Graph-restrictive

Definition

A permutation group L is graph-restrictive if there exists a constant c such that, for every locally-L pair (Γ, G), we have $|G_v| \leq c$.

Example

$\mathrm{Sym}(3)$ (in its natural action) is graph-restrictive, but D_4 is not.

Again, many of the previous results can be proved under the assumption that the local group is graph-restrictive.

What is known?

Conjecture (Weiss 1978)

Primitive groups are graph-restrictive.

Theorem (Weiss, Trofimov 1980-2000)

Transitive groups of prime degree and 2-transitive groups are graph-restrictive.

Theorem (Potočnik, Spiga, V 2012)

Graph-restrictive $\Longrightarrow$ *semiprimitive.*

Theorem (Spiga, V 2014)

Intransitive+graph-restrictive $\Longleftrightarrow$ *semiregular.*

Conjecture (Potočnik, Spiga, V 2012)

Graph-restrictive $\Longleftrightarrow$ *semiprimitive.*

Not graph-restrictive

Theorem (Potočnik, Spiga, V 2015)

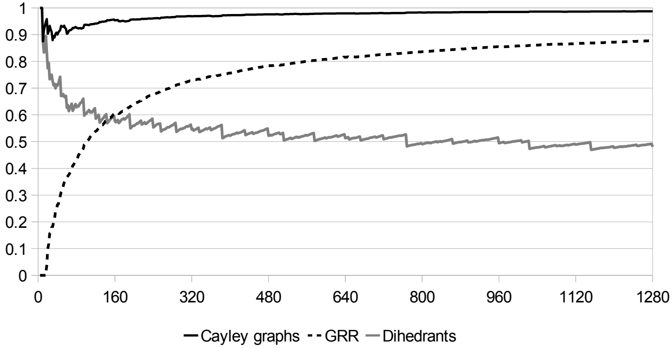
Let (Γ, G) be a locally-D_4 pair. Then one of the following occurs:

1. $\Gamma \cong \mathrm{W}_{m,k}$.
2. $|\mathrm{V}(\Gamma)| \geq 2|G_v|\log_2(|G_v|/2)$.
3. *Finitely other exceptions.*

This is enough to recover some of the results we got in the 3-valent case. For example, enumeration, both asymptotic and small order.

3-valent vertex-transitive

We get a similar result for 3-valent vertex-transitive graphs. In particular, we get a census up to order 1280.



Locally-quasiprimitive

Lemma

Let (Γ, G) be a locally-quasiprimitive pair and let $N \trianglelefteq G$. Then one of the following occurs:

1. *N is semiregular (on vertices of Γ);*
2. *$N_v^{\Gamma(v)}$ is transitive, and N has at most two orbits (on vertices, and two orbits can only occur if Γ is bipartite).*

Proof.

Let N be a non-trivial normal subgroup of G. We have $N_v^{\Gamma(v)}$ is normal in $G_v^{\Gamma(v)}$ which is quasiprimitive, so $N_v^{\Gamma(v)}$ is either trivial, or transitive. In the first case, we get that $N_v = 1$, by a leash argument. In the second case, N is edge-transitive, and the result follows from exercise. □

Polycirculant Conjecture

Conjecture ("Polycirculant Conjecture" Marušič 1981)

Every vertex-transitive (di)graph admits a non-trivial semiregular automorphism.

Known only for a few cases. (Open for graphs of valency 5.)

Theorem (Giudici, Xu 2007)

If $(\Gamma, \mathrm{Aut}(\Gamma))$ *is locally-quasiprimitive, then* $\mathrm{Aut}(\Gamma)$ *contains a non-trivial semiregular element.*

Proof.

By locally-quasiprimitive lemma, can assume $\mathrm{Aut}(\Gamma)$ is quasiprimitive (or bi-quasiprimitive). □

(Includes arc-transitive of prime valency.)

An advertisement for the 41st Australasian Conference on Combinatorial Mathematics and Combinatorial Computing the week 10-14 December 2018 in Rotorua, New Zealand

Exercises about vertex-stabilisers and local actions

1. Prove the basic results for local actions.
2. Prove that primitive groups are quasiprimitive, and quasiprimitive groups are semiprimitive.
3. For each value of $n \geq 3$, determine whether D_n is primitive, quasiprimitive or semiprimitive.
4. (*) Let G be a group generated by a set S of involutions and let $\Gamma = \mathrm{Cay}(G, S)$. Show that if Γ is arc-semiregular, then G is normal in $\mathrm{Aut}(\Gamma)$.
5. (*) (Godsil 1983) : Let G be a 2-group generated by a set S of three involutions. If $\mathrm{Aut}(G, S) = 1$, then $\mathrm{Cay}(G, S)$ is a GRR. (Hint: use the structure of vertex-stabiliser in 3-valent vertex-transitive, and previous exercise.)