8th PhD Summer School in Discrete Maths
Finite Permutation Groups
Lecture 3: Primitive permutation groups

Colva M. Roney-Dougal
colva.roney-dougal@st-andrews.ac.uk

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University
of
St Andrews

§6: Decompositions of permutation groups

Intransitive to transitive

For today: let Ω be finite.

Lemma 32

Let $G \leq \mathrm{Sym}(\Omega)$ be intransitive, and let $\Delta = \alpha^G$. Then the map $G \to \mathrm{Sym}(\Delta)$ that sends each $g \in G$ to the permutation of Δ that it induces is a homomorphism.

Defn: This map is the restriction of G to Δ. Write as $G \to G^\Delta$, $g \mapsto g|_\Delta$.
The image $G^\Delta \leq \mathrm{Sym}(\Delta)$ is a transitive constituent of G.

Abuse notation: think of G^Δ as a subgp of $\mathrm{Sym}(\Omega)$, fixing $\Omega \setminus \Delta$.

Defn: Let $H \leq G_1 \times \cdots \times G_k$. Then H is a subdirect product of $G_1, \ldots, G_k$ if $\forall i \in \underline{k}$, $\forall g_i \in G_i$, $\exists\ h = (h_1, \ldots, h_k) \in H$ s.t. $h_i = g_i$.

Theorem 33

Let $G \leq \mathrm{Sym}(\Omega)$, and let $\Delta_1, \ldots, \Delta_k$ be the orbits of G. Then G is equal to a subdirect product of $G^{\Delta_1} \times \cdots \times G^{\Delta_k}$.

Imprimitive groups embed in wreath products

Theorem 34

Let $G \leq \mathrm{Sym}(\Omega)$ *be imprimitive, with* $\Delta = \Delta_1$ *a block and* $\Sigma = \{\Delta_i \ : \ i \in \underline{n}\} = \{\Delta^g \ : \ g \in G\}$.

Let $K = \{x \in \mathrm{Sym}(\Omega) \ : \ \Delta_i^x \in \Sigma \text{ for } 1 \leq i \leq n\} \leq \mathrm{Sym}(\Omega)$.

Let $\Gamma = \Delta \times \underline{n}$*, and let* $W = \mathrm{Sym}(\Delta) \wr \mathrm{S}_n \leq \mathrm{Sym}(\Gamma)$*, with the imprimitive action.*

Then

1. *K is permutation isomorphic to W: there exists a bijection $\lambda : \Omega \to \Gamma$ and an isom $\psi : K \to W$ s.t. $\forall \alpha \in \Omega$, $k \in K$,*

$$(\alpha^k)\lambda = (\alpha\lambda)^{k\psi}.$$

2. *ψ can be chosen s.t. $G\psi \leq G_\Delta^\Delta \wr G^\Sigma \leq W$.*

Imprimitive to primitive

Previous slide: if G is imprimitive, then G is permutation isomorphic to a subgroup of $G_\Delta^\Delta \wr G^\Sigma$.

If G_Δ^Δ or G^Σ are imprimitive, we may iterate this process.

Summary If G is imprimitive, then G embeds in an iterated wreath product W of primitive groups, in such a way that the actions of each block stabiliser in W, and actions on each system of imprimitivity, are the same as those of G.

Example 35

Let $G = \mathbb{Z}_8$, acting regularly. Then
$\Sigma = \{\Delta_0 = \{0, 4\}, \Delta_1 = \{1, 5\}, \Delta_2 = \{2, 6\}, \Delta_3 = \{3, 7\}\}$
is a system of imprimitivity for G, and G acts as $\mathbb{Z}_4$ on Σ.
Hence $G \leq \mathbb{Z}_2 \wr \mathbb{Z}_4$.
This action of $\mathbb{Z}_4$ is still imprimitive: $\overline{\Delta} = \{\Delta_0, \Delta_2\}$ is a block.
Hence G embeds in $\mathbb{Z}_2 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2$.
Each $\mathbb{Z}_2$ acts primitively.

§7: Normal subgroups of primitive groups

Orbits of normal subgroups

Theorem 36

Let $G \leq \mathrm{Sym}(\Omega)$ *be transitive, and let* $N \trianglelefteq G$*.*
The orbits of N *form a system of imprimitivity for* G*.*

Corollary 37

Let $G \leq \mathrm{Sym}(\Omega)$ *be primitive, and let* $1 \neq N \trianglelefteq G$*.*
Then N *is transitive.*

Minimal normal subgroups

Defn: A minimal normal subgroup of a group $G \neq 1$ is a normal subgroup $1 \neq K \trianglelefteq G$ s.t. if $1 < H < K$ then H is not normal in G.

Theorem 38

G – finite group, $N \trianglelefteq G$.

1. *Let K be a minimal normal subgroup of G. Then either $K \leq N$ or $K \cap N = 1$.*
2. *Every minimal normal subgroup K of G is a direct product of simple groups $T_1 \cong T_2 \cong \cdots \cong T_k$. Each $T_i \trianglelefteq K$ and the T_i are pairwise conjugate in G (so are all isomorphic).*

The socle

Defn: The socle of a group G, written $\mathrm{soc}(G)$, is the subgroup generated by the minimal normal subgroups of G.

Theorem 39

Let $G \le \mathrm{Sym}(\Omega)$ be primitive, with $|\Omega| < \infty$. Let $K \trianglelefteq G$ be minimal normal. Then one of the following holds:

(i) *K is regular and abelian. Then $\mathrm{soc}(G) = K = C_G(K)$.*

(ii) *K is regular and non-abelian, $C_G(K)$ is minimal normal in G, $C_G(K)$ is perm isom to K, and $\mathrm{soc}(G) \cong K \times C_G(K)$.*

(iii) *K is non-abelian, with $C_G(K) = 1$ and $\mathrm{soc}(G) = K$.*

Corollary 40

Let $G \le \mathrm{Sym}(\Omega)$ be primitive, with $|\Omega| < \infty$. Then $\mathrm{soc}(G)$ is a direct product of isomorphic simple groups.

§8: Primitive groups with regular socles

Affine geometries

Let $\mathbb{F}_q$ be the (unique) finite field of order q. Let $V = \mathbb{F}_q^d$.

Defn: The affine geometry $\mathrm{AG}_d(q)$ has:

- points: all vectors in V.
- affine subspaces: all translates of subspaces of V, i.e. all sets

$$S + v = \{u + v \ : u \in S\}, \text{ for } S \leq V, v \in V.$$

An affine automorphism of $\mathrm{AG}_d(q)$ is $\sigma \in \mathrm{Sym}(V)$ that maps affine subspaces to affine subspaces.

The affine general linear group

Let $q = p$ be prime.

Lemma 41
For $a \in \mathrm{GL}_d(p)$ and $v \in \mathbb{F}_p^d$, let $t_{a,v} : \mathbb{F}_p^d \to \mathbb{F}_p^d$, $u \mapsto ua + v$. Then $t_{a,v} \in \mathrm{Aut}(\mathrm{AG}_d(p))$.

Defn: Let $\mathrm{AGL}_d(p) = \{t_{a,v} \ : \ a \in \mathrm{GL}_d(p), v \in \mathbb{F}_p^d\}$ – the affine general linear group.

Theorem 42
Let $G = \mathrm{AGL}_d(p)$.

(i) $G \leq \mathrm{Sym}(p^d)$.

(ii) *Let $V = \mathbb{F}_p^d$. Then $(V, +)$ is permutation isomorphic to $\{t_{1,v} \ : \ v \in V\} \trianglelefteq G$ and V acts regularly.*

(iii) $G \cong V : \mathrm{GL}_d(p) = V : \mathrm{Aut}(V) = V : G_{\underline{0}}$.

Primitive groups of affine type

Defn: $H \leq \mathrm{GL}_d(q)$ is reducible if there exists $0 < U < V = \mathbb{F}_q^d$ s.t. $U^h = U$ for all $h \in H$. Otherwise H is irreducible.

Example 43

The group of all matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ x & a & b \\ y & c & d \end{pmatrix} \in \mathrm{GL}_3(q)$

fixes $\langle(1,0,0)\rangle$, so is reducible.

The group of all matrices of the form $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ y & c & d \end{pmatrix} \in \mathrm{GL}_3(q)$

fixes $\langle(1,0,0),(0,1,0)\rangle$, so is reducible.

$\mathrm{GL}_d(q)$ is irreducible.

Theorem 44

Let $G \leq \mathrm{AGL}_d(p)$, with $(V,+) \trianglelefteq G$.
G is primitive iff $G_{\underline{0}} \leq \mathrm{GL}_d(p)$ is irreducible.

Classification of primitive groups with regular socles

Let $G \leq \mathrm{Sym}(\Omega)$ be primitive, with $|\Omega| < \infty$. Assume that $H = \mathrm{soc}(G)$ is regular.

Theorem 45

Let $N = N_{\mathrm{Sym}(\Omega)}(H)$. Then $N \cong H : \mathrm{Aut}(H) = \mathrm{Hol}(H)$, and $G = H : G_\alpha$.

Theorem 46

If H is abelian, then G is perm isom to some $K \leq \mathrm{AGL}_d(p)$, with $H \cong V \trianglelefteq \mathrm{AGL}_d(p)$. In particular $|\Omega| = p^d$.

Theorem 47

If H is non-abelian then there exists a non-abelian simple T s.t.

1. *$H \cong T_1 \times \cdots \times T_m$ for some m, with $T_i \cong T$;*
2. *G_α acts faithfully and transitively on the T_i, so $G_\alpha \leq \mathrm{S}_m$;*
3. *$N_{G_\alpha}(T_1)$ has a composition factor isomorphic to T, and so T is a comp factor of some $K \leq \mathrm{S}_{m-1}$. In particular, $m \geq 6$.*