8th PhD Summer School in Discrete Mathematics
Vertex-transitive graphs and their local actions I

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Automorphisms of graphs

A (simple) graph Γ is a pair (V, E) with $E \subseteq \binom{V}{2}$. Elements of V are vertices, elements of E are edges.

An automorphism of Γ is a permutation of V that preserves E.

Automorphisms of Γ form $\mathrm{Aut}(\Gamma)$, the automorphism group of Γ.

(Graphs and groups will generally be finite.)

Notation: If $v \in V$ and $g \in \mathrm{Aut}(\Gamma)$, then v^g is the image of v under g.

Vertex-transitive graphs

A graph is vertex-transitive if its automorphism group is transitive (on vertices).

(All vertices identical with respect to the structure of the graph.)

In particular, vertex-transitive $\Longrightarrow$ regular.

Connectedness usually a very mild assumption.

Edge-transitive and arc-transitive graphs

An *s*-arc of a graph is a sequence of $s+1$ vertices $(v_0, \ldots, v_s)$ such that $v_i \sim v_{i+1}$ and $v_i \neq v_{i+2}$.

A 0-arc is a vertex. A 1-arc is just called an arc.

A graph is edge-transitive if its automorphism group acts transitively on edges. (Similarly for arc-transitive, *s*-arc-transitive.)

$(s+1)$-arc-transitive $+$ (minimum valency 2) $\Longrightarrow$ s-arc-transitive

arc-transitive $\Longrightarrow$ edge-transitive

Examples

Γ	Name	Aut(Γ)	Max s	ET?
K_n	Complete	$\mathrm{Sym}(n)$	2	Y
K_n^c	Edgeless	$\mathrm{Sym}(n)$	∞	Y
C_n	Cycle	D_n	∞	Y
$\mathrm{K}_{n,m}$ $m \neq n$	Complete bipartite	$\mathrm{Sym}(n) \times Sym(m)$	X	Y
$\mathrm{K}_{n,n}$	Complete bipartite	$\mathrm{Sym}(n) \wr \mathrm{Sym}(2)$	3	Y
$\mathrm{K}_{m[n]}$	Bal. comp. multip.	$\mathrm{Sym}(n) \wr \mathrm{Sym}(m)$	1	Y
$\mathrm{C}_n \Box \mathrm{K}_2$ $n \neq 4$	Prism	$\mathrm{D}_n \times C_2$	0	N
Q_3	Cube	$C_2 \wr \mathrm{Sym}(3)$	2	Y
Pet	Petersen	$\mathrm{Sym}(5)$	3	Y

Theorem (Weiss 1981)

There is no 8-arc-transitive graph of valency at least 3.

(Relies on Classification of Finite Simple Groups.)

Other definitions and reminders

If G is a transitive subgroup of $\mathrm{Aut}(\Gamma)$, Γ is G-vertex-transitive.

Similarly for G-edge-transitive, etc.

Defined in analogous way: vertex-primitive, G-arc-semiregular, etc.

Lemma (Frattini?)

If G is transitive and $H \leq G$, then $G = HG_v \iff H$ is transitive.

Exercises for Part 1, I

1. Let Γ be G-edge-transitive but not G-vertex-transitive. Show that Γ is bipartite. (Γ is G-bitransitive.)
2. Let Γ be G-edge-transitive and G-vertex-transitive, but not G-arc-transitive. Show that Γ has even valency. (Γ is G-half-arc-transitive.)
3. Start with K_5, then subdivide each edge, then "double" each newly created vertex. Show that the resulting graph has order 20, is regular of valency 4, is edge-transitive but not vertex-transitive. (It is the Folkman Graph.)
4. Show that, if Γ has valency at least 3, there is a largest s such that Γ is s-arc-transitive.

Exercises for Part 1, II

1. Classify all vertex-primitive graphs having two vertices with the same neighbourhood.
2. Let Γ be a connected G-vertex-transitive graph of valency k and let v be a vertex of Γ. Show that there exist k elements $g_1, \ldots g_k$ of G such that $G = \langle G_v, g_1, \ldots, g_k \rangle$.
3. Let Γ be a connected G-arc-transitive graph, let (u, v) be an arc of Γ and let $H = \langle G_u, G_v \rangle$. Prove that $G = H$, unless Γ is bipartite, in which case $|G : H| = 2$.

Cayley graphs

Definition

Let G be a group and $S \subseteq G$. The Cayley graph $\mathrm{Cay}(G, S)$ on G with connection set S has vertex-set G and $u \sim v$ if and only if $uv^{-1} \in S$.

For this to really be a simple graph, we need $1 \notin S$ and

$$S = S^{-1} := \{s^{-1} \mid s \in S\}.$$

The edge-set will be $\{\{g, sg\} \mid g \in G, s \in S\}$.

$\mathrm{Cay}(G, S)$ is connected if and only if $G = \langle S \rangle$.

Let $\tilde{G} \leq \mathrm{Sym}(G)$ be the right regular representation of G.

Lemma

$\tilde{G} \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$.

In particular, Cayley graphs are vertex-transitive.

Examples of Cayley graphs

Γ	G	S	$\mathrm{Aut}(\Gamma)$
C_n	$\mathbb{Z}_n$	$\{-1, +1\}$	D_n
K_n	G_n	G_n^*	$\mathrm{Sym}(n)$
K_n^c	G_n	$\emptyset$	$\mathrm{Sym}(n)$
$\mathrm{K}_{n,n}$	$G_n \times \mathbb{Z}_2$	$G_n \times \{1\}$	$\mathrm{Sym}(n) \wr \mathrm{Sym}(2)$
$\mathrm{K}_{m[n]}$	$G_n \times G_m$	$G_n \times G_m^*$	$\mathrm{Sym}(n) \wr \mathrm{Sym}(m)$
$\mathrm{C}_n \square \mathrm{K}_2$	$\mathbb{Z}_n \times \mathbb{Z}_2$	$\{\pm(1,0), (0,1)\}$	$\mathrm{D}_n \times C_2$
Q_3	$\mathbb{Z}_2^3$	$\{e_1, e_2, e_3\}$	$C_2 \wr \mathrm{Sym}(3)$
Pet	??	??	$\mathrm{Sym}(5)$

Sabidussi's Theorem

Lemma

If Γ is a graph and G is a regular subgroup of $\mathrm{Aut}(\Gamma)$, then $\Gamma \cong \mathrm{Cay}(G, S)$ for some S.

Proof.

Pick a vertex v of Γ, label it with $1 \in G$. For every vertex u of Γ, there is a unique $g \in G$ such that $v^g = u$. Label u with g. Let S be the labels of the neighbourhood of v. Check this works. □

Theorem

$\Gamma \cong \mathrm{Cay}(G, S)$ for some $S \iff \mathrm{Aut}(\Gamma)$ has a regular subgroup isomorphic to G.

If $\mathrm{Aut}(\Gamma)$ is regular, then Γ is called a GRR (graphical regular representation).

Theorem (Godsil)

Most groups admits GRRs. (The exceptions are known.)

Holomorph of a group

Let G be a group. Note that $\mathrm{Aut}(G) \leq \mathrm{Sym}(G)$.

One can check that $\langle \tilde{G}, \mathrm{Aut}(G) \rangle = \tilde{G} \rtimes \mathrm{Aut}(G)$.

This is the holomorph of G.

In fact, $\tilde{G} \rtimes \mathrm{Aut}(G)$ is the normaliser of $\tilde{G}$ in $\mathrm{Sym}(G)$.

Normaliser of $\tilde{G}$ in $\mathrm{Aut}(\Gamma)$

Let G be a group and let $S \subseteq G$. Let $\mathrm{Aut}(G, S)$ be the set of automorphisms of G fixing S.

Lemma

Let $\Gamma = \mathrm{Cay}(G, S)$. Then $\mathrm{Aut}(G, S) \leq \mathrm{Aut}(\Gamma)$. In fact, $\tilde{G} \rtimes \mathrm{Aut}(G, S)$ is the normaliser of $\tilde{G}$ in $\mathrm{Aut}(\Gamma)$.

Proof.

The first part is an easy calculation. Next, note that the normaliser of $\tilde{G}$ must be contained in $\tilde{G} \rtimes \mathrm{Aut}(G)$ but any element of $\mathrm{Aut}(G)$ fixes the identity so fixes its neighbourhood S. □

Examples, revisited

Γ	G	S	$\mathrm{Aut}(G, S)$	$\tilde{G} \rtimes \mathrm{Aut}(G, S)$
C_n	$\mathbb{Z}_n$	$\{-1, +1\}$	$\overline{-1}$	D_n
K_n	G_n	G_n^*	$\mathrm{Aut}(G)$	$\mathrm{Hol}(G_n)$
$\mathrm{C}_n \Box \mathrm{K}_2$	$\mathbb{Z}_n \times \mathbb{Z}_2$	$\{\pm(1, 0), (0, 1)\}$	$\overline{-1} \times 1$	$\mathrm{D}_n \times \mathrm{C}_2$
$n \geq 3$				
Q_3	$\mathbb{Z}_2^3$	$\{e_1, e_2, e_3\}$	$\mathrm{Sym}(3)$	$\mathrm{C}_2 \wr \mathrm{Sym}(3)$

If $\mathrm{Aut}(\mathrm{Cay}(G, S)) = \tilde{G} \rtimes \mathrm{Aut}(G, S)$, then $\mathrm{Cay}(G, S)$ is a normal Cayley graph.

Exercises on Cayley graphs

1. Complete the proofs of basic facts about Cayley graphs. (Connectedness, Sabidussi's Theorem, normaliser of $\tilde{G}$.)
2. Prove that a vertex-transitive graph of prime order is Cayley.
3. Prove that a Cayley graph of valency at least 3 on an abelian group has girth at most 4.
4. Let G be an abelian group with an element of order at least 3. Prove that G does not admit a GRR.
5. Show that the Petersen graph is not a Cayley graph. (You may assume that $\mathrm{Aut}(Pet) \cong \mathrm{Sym}(5)$.)
6. For what values of n is K_n a normal Cayley graph?
7. (*) Show that an edge-transitive Cayley graph on an abelian group is arc-transitive.