8th PhD Summer School in Discrete Mathematics
Questions on Colva's lectures on July 4th.

1. Let G be a group, let K be a minimal normal subgroup of G, and let $N \trianglelefteq G$. Show that either $K \leq N$ or $K \cap N = 1$.

2. Let $G = T_1 \times \cdots \times T_m$ is a direct product of m nonabelian simple groups T_i. Show that the T_i are the only minimal normal subgroups of T.

3. Let T be a nonabelian simple group, and let $k \geq 1$ show that

$$\mathrm{Aut}(T^k) \cong \mathrm{Aut}(T) \wr \mathrm{S}_k.$$

4. Let G be a group, and let $N \trianglelefteq G$. Show that $C_G(N) \trianglelefteq G$.

5. The following theorem was omitted from my lectures:
Let K be a transitive subgroup of $\mathrm{Sym}(\Omega)$, and let $C = C_{\mathrm{Sym}(\Omega)}(K)$.
Then (i) $C_\alpha = 1$ for all $\alpha \in \Omega$; (ii) the group C is transitive if and only if K is regular; and (iii) if C is transitive, then $C = K^\sigma$ for some $\sigma \in \mathrm{Sym}(\Omega)$.
Use this theorem to prove the following:

(a) Let $G \leq \mathrm{Sym}(\Omega)$ be primitive, and let K be a minimal normal subgroup of G. Let $C = C_G(K)$. Then either $C = 1$ or C is permutation isomorphic to K.

(b) Hence show that $\mathrm{soc}(G) = KC$.

(c) Hence prove Theorem 39.

6. An *affine basis* for $\mathrm{AG}_d(q)$ is a set $B = \{v_0, \ldots, v_d\}$ of $d+1$ vectors from $\mathbb{F}_q^d$ such that B is not contained in any $(d-1)$-dimensional affine subspace. Show that $\mathrm{AGL}_d(q)$ acts *regularly* on the set of affine bases of $\mathrm{AG}_d(F)$.

7. Show that $\mathrm{AGL}_d(p)$ is 2-transitive, and that $\mathrm{AGL}_d(2)$ is *3-transitive*: given any two triples of points $(u_1, u_2, u_3), (v_1, v_2, v_3) \in \mathbb{F}_p^3$, with $u_i \neq u_j$ for $i \neq j$ and $v_i \neq v_j$ for $i \neq j$, there exists a $t_{a,v} \in \mathrm{AGL}_d(2)$ s.t. $u_i^{t_{a,v}} = v_i$ for $1 \leq i \leq 3$.

8. Let H be a regular subgroup of $\mathrm{Sym}(\Omega)$, and let

$$N = N_{\mathrm{Sym}(\Omega)}(H) = \{\sigma \in \mathrm{Sym}(\Omega) \ : \ h^\sigma \in H \text{ for all } h \in H\}.$$

(a) Show that $N = H \ : \ N_\alpha$.

(b) It follows that N_α acts on H by conjugation, so there is a homomorphism $\phi : N_\alpha \to \mathrm{Aut}(H)$. Show that $\mathrm{Im}\phi = \mathrm{Aut}(H)$.

(c) Show that the map ϕ is injective.