8th PhD Summer School in Discrete Mathematics
Questions on Colva's lectures on July 3rd.

1. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

2. Prove that $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$, for all primes p.

3. Let D_4 denote the dihedral group on 4 points. Prove that $\mathrm{Aut}(D_4) \cong D_4$.

4. Let K be a group. Show that we can define an action of $K \times K$ on K by $a^{(x,y)} = x^{-1}ay$ for all $a \in K$ and $(x, y) \in K \times K$. Show that this action is transitive, and find the stabiliser of 1_K. When is the action faithful?

5. Let $N \trianglelefteq G$ and $M \trianglelefteq G$. Show that if $N \cap M = 1$ then $nm = mn$ for all $m \in M$, $n \in N$. Prove that $G_1 \times G_2$ is a semidirect product of G_1 and G_2.

6. Show that every dihedral group is a semidirect product of two cyclic groups. Show that the quaternion group Q_8 may not be decomposed (nontrivially) as a semidirect product.

7.
(i) Let $G = N : H$, and consider the action of G on the right cosets of H. Show that the image of N in this representation is a regular permutation group.
(ii) Let $G \leq \mathrm{Sym}(\Omega)$ and $\alpha \in \Omega$. Assume that G has a regular normal subgroup K. Show that G is a semidirect product of K and G_α.

8. Let G act on Ω and let $\Delta \subset \Omega$. Prove the following:
(i) $G_{\{\Delta\}} \leq G$;
(ii) if G is transitive and Δ is a block for G, then $G_{\{\Delta\}}$ is transitive on Δ.

9. Suppose that G acts on a set Ω, with the property that for any two ordered pairs $(\alpha, \beta), (\gamma, \delta) \in \Omega^2$ with $\alpha \neq \beta$ and $\gamma \neq \delta$ there exists $g \in G$ s.t. $\alpha^x = \gamma$ and $\beta^x = \delta$. Such a group is called *2-transitive*. Show that G is primitive.

10. Let F be a field, and let $G \leq \mathrm{Sym}(F)$ consist of all permutations of the form $\epsilon \mapsto \epsilon\alpha + \beta$, with $\alpha \in F \setminus \{0\}$ and $\beta \in F$. Show that G is 2-transitive.

11. Let $G = \mathrm{S}_k \wr \mathrm{S}_2 \leq \mathrm{S}_{2k}$ (so G is acting imprimitively, with blocks $\{1, \ldots, k\}$ and $\{k+1, \ldots, 2k\}$, say). Show that G is a maximal subgroup of S_{2k}.
[Hint: Show that if $g \notin G$ then $\langle G, g\rangle$ contains all transpositions]