Combinatorial limits and their applications in extremal combinatorics

Part 4

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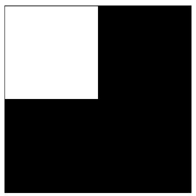
Flag algebras

- algebra $\mathcal{A}$ of formal linear combinations of graphs
  addition and multiplication by a scalar
- homomorphism $f_W : \mathcal{A} \to \mathbb{R}$ for a graphon W
  $f_W(\sum \alpha_i H_i) := \sum \alpha_i d(H_i, W)$
  multiplication, elements always in $\mathrm{Ker}(f_W)$
- algebra $\mathcal{A}^R$ of R-rooted graphs
  random homomorphism $f_W^R : \mathcal{A}^R \to \mathbb{R}$



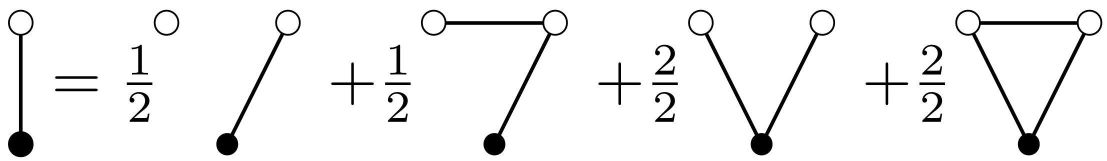
$$\frac{f^\bullet(K_2^\bullet) = 1/2,\ f^\bullet(\overline{K_2^\bullet}) = 1/2,\ f^\bullet(K_3^\bullet) = 1/4,\ \ldots}{f^\bullet(K_2^\bullet) = 1,\ f^\bullet(\overline{K_2^\bullet}) = 0,\ f^\bullet(K_3^\bullet) = 3/4,\ \ldots}$$

Questions?

Operations with rooted graphs

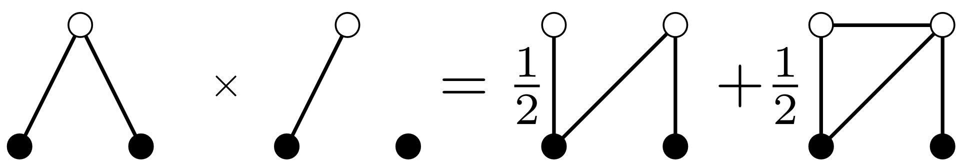
- projection

  prob. that deleting non-root vertices yields the flag



- multiplication

  prob. partitioning non-root vertices yields the terms



Expected value

- goal: $\mathbb{E}_R f_W^R(H) = f_W(\llbracket H \rrbracket_R)$ for $H \in \mathcal{A}^R$
- $f(\llbracket H \rrbracket_\bullet) = \mathbb{E}_z f^z(H)$

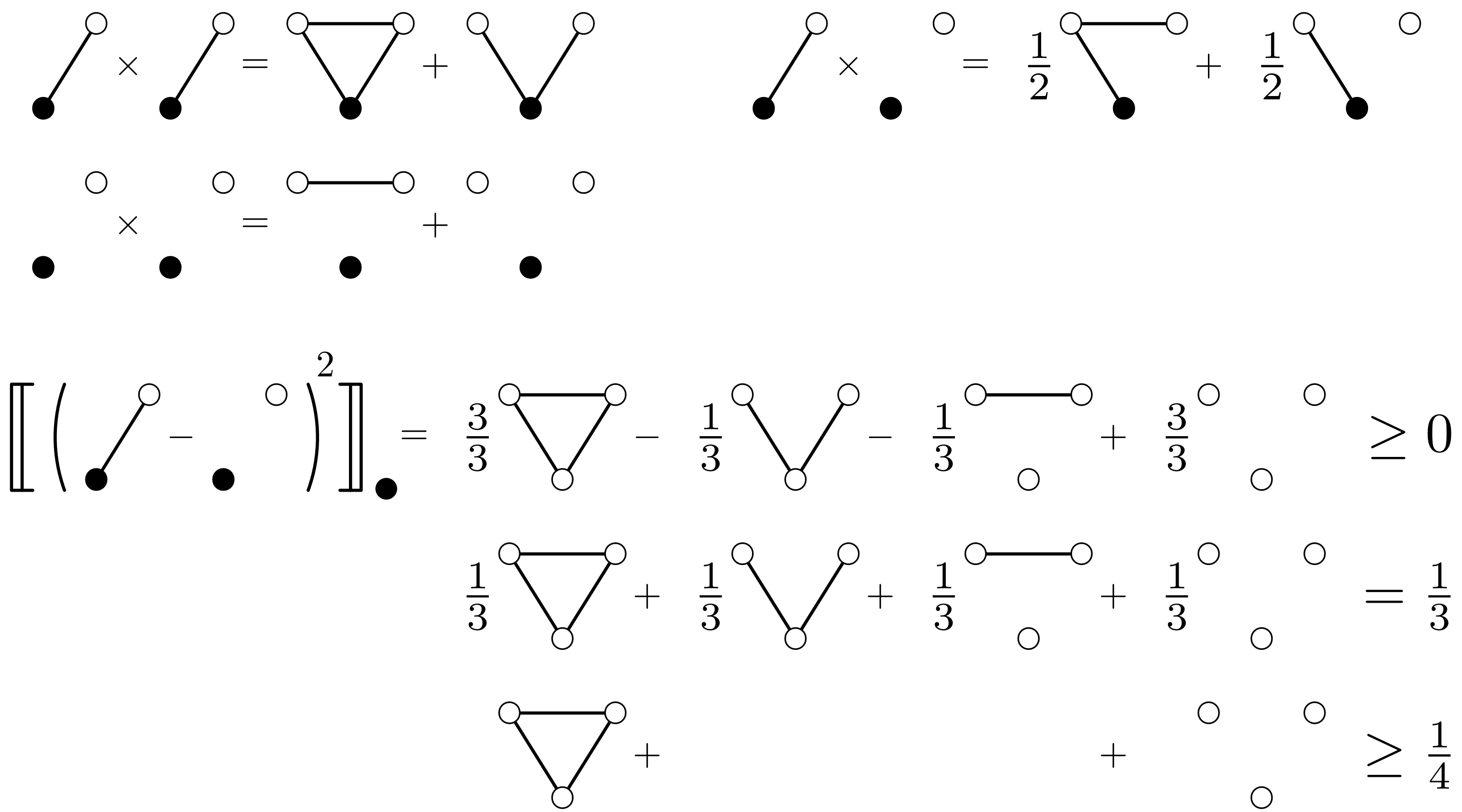
- $\llbracket \cdot \rrbracket_R : \mathcal{A}^R \to \mathcal{A} \qquad \llbracket H \rrbracket_R = \alpha H'$
  H' is the graph H without distinguishing roots
  α is the prob. that randomly chosen roots yield H

Questions?

Computing with flags

- simple applications yields results such as
  $f_W(K_2) > 1/2 \Rightarrow f_W(K_3) > 0$ for every W
  $f_W(K_3 + \overline{K_3}) \geq 1/4$ for every W
- shorthand notation for $x, y \in \mathcal{A}$
  $x = y \Leftrightarrow \forall W \; f_W(x) = f_W(y)$
  $x \geq 0 \Leftrightarrow \forall W \; f_W(x) \geq 0$
- What can we use in computations?
  $x^2 \geq 0$ for every $x \in \mathcal{A}$
  $[\![x^2]\!]_R \geq 0$ for every $x \in \mathcal{A}^R$

Goodman's theorem



Questions?

Flag algebras

- algebra $\mathcal{A}$ of formal linear combinations of graphs
- homomorphism $f_W : \mathcal{A} \to \mathbb{R}$ for a graphon W
  $f_W(\sum \alpha_i H_i) := \sum \alpha_i d(H_i, W)$
  multiplication, other relations between elements
- algebra $\mathcal{A}^R$ of R-rooted graphs
  random homomorphism $f_W^R : \mathcal{A}^R \to \mathbb{R}$
  multiplication, average operator $[\![\cdot]\!]_R : \mathcal{A}^R \to \mathcal{A}$
  $\mathbb{E}_R f_W^R(x) = f_W([\![x]\!]_R)$ for every $x \in \mathcal{A}^R$
- $f_W([\![x^2]\!]_R) \geq 0$ - how to find suitable x?

SDP FORMULATION

- find maximum α_0 such that $f_W(G_0) \geq \alpha_0$ assming $f_W(G_i) \geq \alpha_i$ where $G_0, \ldots, G_k \in \mathcal{A}$
- What inequalities can we use?
  $f_W(G') \geq 0$ for any graph G'
  $f_W(K_1) = 1$ where K_1 expressed in n-vertex graphs
  $f_W(\llbracket x^2 \rrbracket_R) \geq 0$ for $x \in \mathcal{A}^R$
- let $H_1, \ldots, H_m$ be elements of $\mathcal{A}^R$, $h = (H_1, \ldots, H_m)$
  if $M \succeq 0$, then $f_W(\llbracket h^T M h \rrbracket_R) \geq 0$

SDP formulation

- prove $f_W(G_0) \geq \alpha_0$ assuming $f_W(G_i) \geq \alpha_i$

- find $\gamma_i \geq 0,\ \delta_0 \in \mathbb{R},\ \delta_i \geq 0,\ M \succeq 0$
  $G_0 = \sum_{i=1}^{k} \gamma_i G_i + \sum_{i=1}^{\ell} (\delta_0 + \delta_i) G'_i + [\![h^T M h]\!]_R$
  $\alpha_0 = \delta_0 + \sum_{i=1}^{k} \gamma_i \alpha_i$
  where $G'_1, \ldots, G'_\ell$ are all n-vert. graphs and $h \in (\mathcal{A}^R)^m$

- $\gamma_i \times f_W(G_i) \geq \gamma_i \times \alpha_i$
  $\delta_0 \times f_W(G'_1 + \cdots + G'_\ell) = \delta_0 \times 1$
  $\delta_i \times f_W(G'_i) \geq 0$
  $f_W([\![h^T M h]\!]_R) \geq 0$

SDP EXAMPLE

- prove $f_W(\overline{K_3} + K_3) \geq \alpha_0$ for maximum α_0
- $(G'_1, \ldots, G'_4) = (\overline{K_3}, \overline{K_{1,2}}, K_{1,2}, K_3)$, $h = (\overline{K_2}^{\bullet}, K_2^{\bullet})$
- SDP: $\max\langle C, X\rangle$ s.t. $\langle A_i, X\rangle = b_i$, $X \succeq 0$, $X \in \mathbb{R}^{8\times 8}$

$$C = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad X = \begin{matrix} 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & -3/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3/4 & 3/4 \end{matrix}$$

$$A_1 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad A_2 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \end{matrix}$$

$$b_1 = 1 \qquad b_2 = 0$$

$$A_3 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{matrix} \qquad A_4 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$b_3 = 0 \qquad b_4 = 1$$

SDP FORMULATION

- prove $f_W(G_0) \geq \alpha_0$ if $f_W(G_i) \geq \alpha_i$
- find $\gamma_i \geq 0$, $\delta_0 \in \mathbb{R}$, $\delta_i \geq 0$, $M \succeq 0$
  $G_0 = \sum_{i=1}^{k} \gamma_i G_i + \sum_{i=1}^{\ell} (\delta_0 + \delta_i) G_i' + [\![h^T M h]\!]_R$
  $\alpha_0 = \delta_0 + \sum_{i=1}^{k} \gamma_i \alpha_i$
  where $G_1', \ldots, G_\ell'$ are all n-vert. graphs and $h \in (\mathcal{A}^R)^m$
- SDP: $\max \langle C, X \rangle$ s.t. $\langle A_i, X \rangle = b_i$ and $X \succeq 0$
  X of size $k + 2 + \ell + m$, diagonal γ_i, $\pm\delta_0$, δ_i, M
  ℓ constraints, b_i is the coefficient of G_i' in G_0

Questions?

Thank you for your attention!