An infinite-dimensional $\square_q$-module obtained from the q-shuffle algebra for affine $\mathfrak{sl}_2$

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Abstract

Let $\mathbb{F}$ denote a field, and pick a nonzero $q \in \mathbb{F}$ that is not a root of unity. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4. Define a unital associative $\mathbb{F}$-algebra $\square_q$ by generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

$$\frac{q x_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = 1,$$
$$x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0,$$

where $[3]_q = (q^3 - q^{-3})/(q - q^{-1})$. We will review how $\square_q$ is related to the q-Onsager algebra. We will review the classification of the finite-dimensional irreducible $\square_q$-modules, and how these modules give an example of a tridiagonal pair. Our new results concern a set of infinite-dimensional $\square_q$-modules, said to be NIL. Let W denote a $\square_q$-module. A vector $\xi \in W$ is called NIL whenever $x_1 \xi = 0$ and $x_3 \xi = 0$ and $\xi \neq 0$. The $\square_q$-module W is called NIL whenever W is generated by a NIL vector.

We show that up to isomorphism there exists a unique NIL $\square_q$-module, and it is irreducible and infinite-dimensional. We describe this module from sixteen points of view. In this description an important role is played by the q-shuffle algebra for affine $\mathfrak{sl}_2$. This is joint work with Sarah Post.