Lifting techniques in covering graphs and applications

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1. Covering graphs

A Covering **from a graph** X **to a graph** Y**:** $\exists$ **a surjective** $p : V(X) \to V(Y)$**, s. t. if** $p(x) = y$ **then** $p|_{N(x)} : N(x) \to N(y)$ **is a bijection**

X**: covering graph:** Y**: base graph;**

Vertex fibre: $p^{-1}(v)$**,** $v \in V(Y)$**; Edge fibre:** $p^{-1}(e)$**,** $e \in E(Y)$**;**

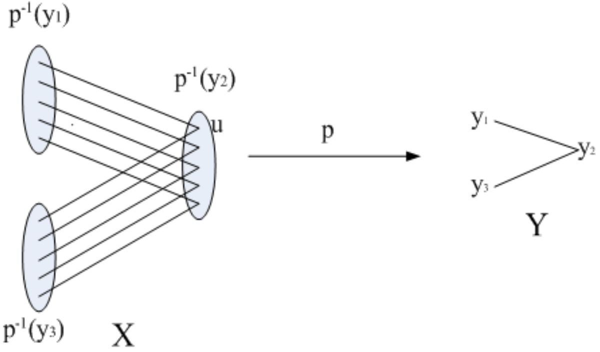
G : **the group of fibre-preserving automorphisms of** X

Covering transformation group K**: the kernel of** G **acting on the fibres.**

X **is connected** $\Longrightarrow$ K **acts semiregualrly on each fibre.**

Regular Cover: K **acts regularly on each fibre.**

$K \lhd G$**,** $\quad G/K \leq \textbf{Aut}\,(Y)$**.**



p-1(y1)
p-1(y2)
u
p-1(y3)
X
p
y1
y2
y3
Y

Voltage graphs

Gross and Tucker (1974).

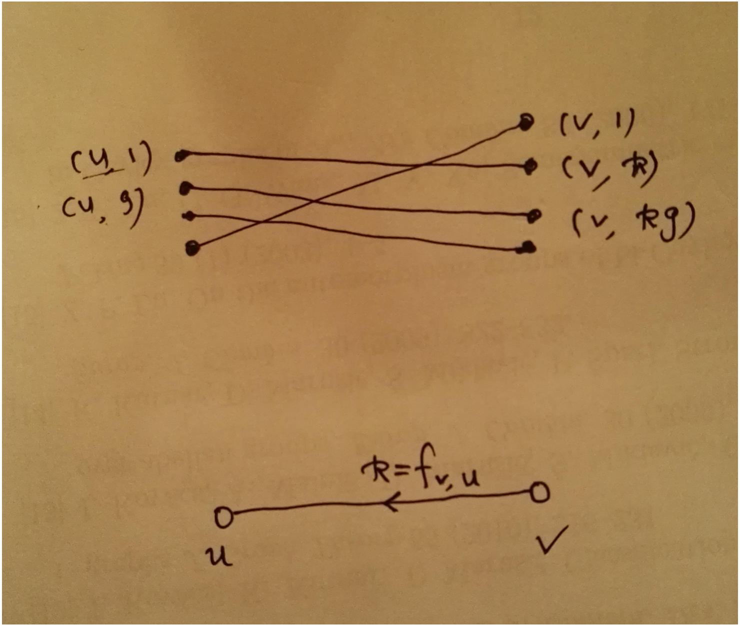
J.L. Gross and T.W. Tucker, *Topological Graph Theory,* **Wiley, New York, 1987.**

Voltage assignment f**: graph** Y**, finite group** K
a function $f : A(Y) \to K$ **s. t.** $f_{u,v} = {f_{v,u}}^{-1}$ **for each** $(u, v) \in A(Y)$.

Voltage graph: (Y, f)

Derived graph $Y \times_f K$:

vertex set $V(Y) \times K$,
arc-set $\{((u, g), (v, f_{v,u}g) \mid (u, v) \in A(Y),\ g \in K\}$.


(u, 1)
(u, g)
(v, 1)
(v, k)
(v, kg)
k=f_{v,u}
u
v

Lifting: $\alpha \in \mathbf{Aut}\,(Y)$ **lifts to an automorphism** $\overline{\alpha} \in \mathbf{Aut}\,(X)$ **if** $\overline{\alpha}p = p\alpha$.

$$\begin{array}{ccc} & \overline{\alpha} & \\ X & \rightarrow & X \\ p\downarrow & & \downarrow p \\ Y & \rightarrow & Y \\ & \alpha & \end{array}$$

General Question:

Given a graph Y, a group K and $H \leq \mathbf{Aut}(Y)$, find all the connected regular coverings $Y \times_f K$ on which H lifts.

Note : if H lifts to G, then $G/K \cong H$.

A lifting problem is essentially a group extension problem

$1 \to K \to G \to H$

Lifting Theorem

Lifting Theorem: let $X = Y_f \times K$**,** $\alpha \in \mathbf{Aut}(Y)$**. Then**

α **lifts if and only if** $f_{W^\alpha} = 1$ **is equivalent to** $f_W = 1$**,**

for each closed walk W **in** Y.

A. Malnič, Group actions, coverings and lifts of automorphisms, Discrete Math. 182 (1998), 203-218.

Theorem: Let $X = Y \times_f K$ be a connected regular cover of a graph Y, where K is abelian, If $\alpha \in$ Aut Y is an automorphism one of whose liftings $\tilde{\alpha}$ centralizes K, then $f_{W^\alpha} = f_W$ for any closed W of Y.

S.F. Du, J.H.Kwak and M.Y.Xu, On 2-arc-transitive covers of complete graphs with covering transformation group Z_p^3, J. Combin. Theory, B 93 (2005), 73–93.

Elementary abelian covering group $K = Z_p^n$.

Aleksander Malnic, Primoz Potocnik, Invariant subspaces, duality, and covers of the Petersen graph, European J. Combin. 27 (2006), no. 6, 971(989)

S.F. Du, J.H. Kwak and M.Y. Xu, Linear criteria for lifting of automorphisms in elementary abelian regular coverings, Linear Alegebra and Its Applications, 373, 101-119(2003).

Abelian covers:

1. Conder, Ma, Arc-transitive abelian regular covers of the Heawood graph. *J. Algebra* **387 (2013), 243-267.**

2. Conder, Ma, Arc-transitive abelian regular covers of cubic graphs. *J. Algebra* **387 (2013), 215-242.**

2. Relationship between topological lifting theorem and group extensions

Sabidussi Coset graph:

given a group G **,** $H \leq G$**,** $a \in G$**, s. t.** $HaH = Ha^{-1}H$**,** $\langle H, a \rangle = G$**.**

Define a graph $Cos(G, H, HaH)$**:**

Vertex set $\{Hg \mid g \in G\}$**,** **Edge set** $\{H, Ha\}^G$**.**

Note **Every arc transitive graph can be represented by a Coset graph.**

A Coset graph gives more information of groups

A voltage graph gives more clearly and simple adjacent relations, but the properties of the groups are hidden

For some small graphs, lifting theorem can be only used to determine the voltage assignment.

For most cases, group theoretical method (the coset graphs) may be applied to determine the covering graph.

Combining voltage graph, lifting theorem, group extension together, one may work on more complicate and deep cases.

General idea from group theory:

to classify the regular covers of Y having ctg K with a symmetric property (*)

1: find all the some subgroups $H \leq \mathbf{Aut}(Y)$, insuring this (*)

2: determine the group extension $1 \to K \to G \to H$

3: determine coset graphs from G

Three possibilities:

(1) There exists such classification for H and also it is feasible to determine the extension $1 \to K \to G \to H$

(2) we do have such classification for H but it is very complicated and almost infeasible to determine the extension

(3) we cannot have such classification for H.

New Idea:

1: Instead of using the classification of H, choose a subgroup H_1 of H, so that we may determine the extension ($G_1/K = H_1$), where H_1 does not need to insure $(*)$

2: find all Coset graphs from G_1, from which we find the voltage graphs X (the voltage assignment is very simple and nice; with high symmetric properties *, there are not so many such X)

3. for the above X, choose a subgroup $H_2 \geq H_1$ which insuring (*), then use Lifting Theorem to show H_2 lifts.

Problem: $Y = K_5$, $V = \{0, 1, 2, 3, 4\}$, $K = (V(3, q), +)$

Find all regular covers $K_5 \times_f K$ of Y such that A_5 lifts.

Solution: $X(p) = K_5 \times_f K$, **where**

$f_{0,j} = (0, 0, 0)$ **for** $1 \leq j \leq 4$, $f_{1,2} = (1, 0, 0)$, $f_{1,3} = (0, 1, 0)$, $f_{2,3} = (0, 0, 1)$, $f_{1,4} = (a, b, c)$, $f_{2,4} = (-b, -c, a)$ **and** $f_{3,4} = (c, -a, -b)$, **where** $a = \frac{1+\sqrt{5}}{4}$, $b = \frac{1-\sqrt{5}}{4}$ **and** $c = \frac{\sqrt{5}}{2}$.

where either $q = 5$ **or** $q = \pm 1 (\mod 10)$.

$G/\mathbb{Z}_p^3 = A_5$.

first we need to use the (ordinary and modular) representations of dimension 3 of A_5 to determine G, then compute coset graphs.

Lemma

X must be isomorphic to $X(p)$.

Proof **Take a basis** $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ **in** $K = V(3, p)$**.**

Take a spanning tree Y_0 **of** K_5 **with root** 0

assume that $f_{0,i} = \mathbf{0}$ **for any** $i \in V_1 := \{1, 2, 3, 4\}$.

Case (1) In $K_5[V_1]$, the three voltages on the respective arcs in any triangle are linearly dependent, but there exists a claw such that the three voltages on the arcs in this claw are linearly independent.

Assume that $f_{1,2} = \mathbf{x},\ f_{1,3} = \mathbf{y},\ f_{1,4} = \mathbf{z}$ and $f_{2,3} = a\mathbf{x} + b\mathbf{y}$.

Take a closed walk $W = ((0,1,2)^a, (0,1,3)^b, 0, 3, 2, 0)$, where

$$f_W = af_{1,2} + bf_{1,3} - f_{2,3} + (a+b)f_{0,1} + (1-a)f_{0,2} - (b-1)f_{0,3} = \mathbf{0}.$$

Take $(243) \in A_5$. Then

$$f_{W^{(243)}} = af_{1,4} + bf_{1,2} - f_{4,2} + (a+b)f_{0,1} + (1-a)f_{0,4} - (b-1)f_{0,2} = \mathbf{0},$$

So we have $f_{2,4} = -b\mathbf{x} - a\mathbf{z}$.

Since $f_{W^{(12)(34)}} = \mathbf{0}$ and $f_{W^{(012)}} = \mathbf{0}$ respectively, we have

$$-a\mathbf{x} - \mathbf{z} + bf_{2,4} = \mathbf{0} \quad \text{and} \quad (a+b)\mathbf{x} + (1-b)\mathbf{y} + bf_{2,3} = \mathbf{0}. \ (3.1)$$

Substituting the values of $f_{2,3}$ and $f_{2,4}$ in (3.1), we get

$$a + b^2 = 0, \ 1 + ab = 0 \quad \text{and} \quad a + b + ab = 0.$$

Check: it has no solutions.

Case (2) In $K_5[V_1]$, there exists a triangle such that three voltages assigned to its arcs are linearly independent.

Assume that $f_{1,2} = \mathbf{x},\ f_{1,3} = \mathbf{y},\ f_{2,3} = \mathbf{z}$ and $f_{1,4} = a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$.

Take a closed walk

$$W = ((0,1,2)^a, (0,1,3)^b, (0,2,3)^c, 0, 4, 1, 0),$$

where

$$f_W = af_{1,2} + bf_{1,3} + cf_{2,3} - f_{1,4} + (a+b-1)f_{0,1} + (c-a)f_{0,2} - (b+c)f_{0,3} + f_{0,4} = \mathbf{0}.$$

Inserting $(132), (123), (12)(34), (02)(13)$**, we get**

$$f_{3,4} = c\mathbf{x} - a\mathbf{y} - b\mathbf{z}, f_{2,4} = -b\mathbf{x} - c\mathbf{y} + a\mathbf{z}.$$

$$-a + ac - b^2 = 0, -1 + c^2 + ab = 0, a - b - c = 0, 2a + 2b - 1 = 0.$$

Solving this equation system, we get
$f_{0,j} = (0, 0, 0)$ **for** $1 \leq j \leq 4$, $f_{1,2} = (1, 0, 0)$**,** $f_{1,3} = (0, 1, 0)$**,**
$f_{2,3} = (0, 0, 1)$**,** $f_{1,4} = (a, b, c)$, $f_{2,4} = (-b, -c, a)$ **and**
$f_{3,4} = (c, -a, -b)$**, where** $a = \frac{1+\sqrt{5}}{4}$**,** $b = \frac{1-\sqrt{5}}{4}$ **and** $c = \frac{\sqrt{5}}{2}$**.**

where either $q = 5$ **or** $q = \pm 1 \pmod{10}$. $\square$

Lemma

Show A_5 lifts

Proof $A_5 = \langle (13)(24), (012) \rangle$.

Let W **be a closed walk in** K_5 **with** $f_W = \mathbf{0}$. **We may assume that the arc** (i, j) **(resp.** (j, i)**) appears** $\ell_{i,j}$ **(resp.** $\ell_{j,i}$**) times in** W **and let** $t_{i,j} = \ell_{i,j} - \ell_{j,i}$. **Since** $f_{i,j} = -f_{j,i}$, **we get** $t_{i,j} = -t_{j,i}$. **Then** $f_W = \sum_{0 \le i < j \le 4} t_{i,j} f_{i,j} = \mathbf{0}$.

Substituting the values of $f_{i,j}$ in it, we get the following three relations between $\{t_{i,j}\}$;

$$\begin{aligned} t_{1,2} &= -at_{1,4} + bt_{2,4} - ct_{3,4}, \\ t_{1,3} &= -bt_{1,4} + ct_{2,4} + at_{3,4}, \\ t_{2,3} &= -ct_{1,4} - at_{2,4} + bt_{3,4}. \end{aligned} \tag{3.3}$$

Since W is a closed walk, the numbers of arcs in W coming from i and going into i are equal for any vertex i in $V(K_5)$. So we get

$$\begin{aligned} t_{0,1} &= t_{1,2} + t_{1,3} + t_{1,4} = (1-a-b)t_{1,4} + (b+c)t_{2,4} + (a-c)t_{3,4}, \\ t_{0,2} &= t_{2,1} + t_{2,3} + t_{2,4} = (a-c)t_{1,4} + (1-a-b)t_{2,4} + (c+b)t_{3,4}, \\ t_{0,3} &= t_{3,1} + t_{3,2} + t_{3,4} = (b+c)t_{1,4} + (a-c)t_{2,4} + (1-a-b)t_{3,4}, \\ t_{0,4} &= t_{4,1} + t_{4,2} + t_{4,3} = -t_{1,4} - t_{2,4} - t_{3,4}. \end{aligned} \tag{3.4}$$

Let $\alpha = (13)(24)$. **Then**

$$f_{W^\alpha} = \sum_{0 \le i < j \le 4} t_{i,j} f_{i^\alpha, j^\alpha}$$

$= \mathbf{t}_{1,2} f_{3,4} + t_{1,3} f_{3,1} + t_{2,3} f_{4,1} + t_{1,4} f_{3,2} + t_{2,4} f_{4,2} + t_{3,4} f_{1,2}$.
Substituting the values of $f_{i,j}$ **in it and by using (3.3), we get**

$$\begin{aligned} f_{W^\alpha} &= (ct_{1,2} - at_{2,3} + bt_{2,4} + t_{3,4})\mathbf{x} \\ &\quad + (-at_{1,2} - t_{1,3} - bt_{2,3} + ct_{2,4})\mathbf{y} \\ &\quad + (-bt_{1,2} - ct_{2,3} - t_{1,4} - at_{2,4})\mathbf{z} \\ &= \left((bc + a^2 + b)t_{2,4} - (c^2 + ab - 1)t_{3,4}\right)\mathbf{x} \\ &\quad + \left((a^2 + b + bc)t_{1,4} + (ac - a - b^2)t_{3,4}\right)\mathbf{y} \\ &\quad + \left(-(1 - ab - c^2)t_{1,4} + (-b^2 + ac - a)t_{2,4}\right)\mathbf{z}. \end{aligned} \tag{3.5}$$

Since $(a, b, c) = (\frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}, \frac{\sqrt{5}}{2})$**, it is easy to check that** $bc + a^2 + b = c^2 + ab - 1 = ac - b - b^2 = 0$. **Hence** $f_{W^\alpha} = \mathbf{0}$ **and so** α **lifts.**

Let $\beta = (012)$. **Similarly, show** $f_{W^\beta} = \mathbf{0}$. **So** A_5 **lifts.**

Question: $Y = K_{n,n}$, $K = \mathbb{Z}_p^2$, **find the covers** $X = Y_f \times K$ **such that the fibre-preserving subgroup acts 2-arc-transitively**

$V(Y) = U \cup W$, $\mathbf{Aut}\,(Y) = (S_n \times S_n) \rtimes Z_2$.

A : **a** 2-**arc-transitive subgroup of** $\mathbf{Aut}\,(Y)$, $G = A_U = A_W$

$\widetilde{A}/K \cong A$, $\widetilde{G}/K \cong G$

G_u **acts 2-tran. on** $W \Longrightarrow G$ **is a 2-transitive group of** X **on** W **and so on** U.

Affine case: $G^U \leq \mathrm{AGL}(s,p) = \mathbb{Z}_p^s \rtimes \mathrm{GL}(s,p)$

$Y = K_{p^s,p^s}$, $s \geq 2$, $K = \mathbb{Z}_p^2$, $V = U \cup W$

$$U = \{\alpha \mid \alpha \in V(s,p)\},\ W = \{\alpha' \mid \alpha \in V(s,p)\},$$

$T_U \cong T_W \cong \mathbb{Z}_p^s$, $T = T_U \times T_W$.

$\widetilde{T}_U/K = T_U$, $\widetilde{T}_W/K = T_W$, $\widetilde{T}/K = (T_U \times T_W)/K$.

$G = (T_U \times T_W) \rtimes H$, $H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p)$

$A = G\langle\sigma\rangle$, σ **exchanges two biparts of** $K_{n,n}$.

Group problem:

$\widetilde{G}/\mathbb{Z}_p^2 = (\mathbb{Z}_p^s \times \mathbb{Z}_p^s) \rtimes H$, $H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p)$,

where H **is tran on** $V(s,p) \setminus \{0\}$.

Group theoretical approach:

1. **Determine** p**-subgroups** P **of** $\widetilde{G}$ **such that** $P/\mathbb{Z}_p^2 = \mathbb{Z}_p^s \times \mathbb{Z}_p^s$;
2. **Determine** $\widetilde{G} = P.\widetilde{H}$, **where** $\widetilde{H}/K = H$.

$P/\mathbb{Z}_p^2 = \mathbb{Z}_p^s \times \mathbb{Z}_p^s$.

$c = 2$, $\exp(P) = p$, $Z(P) = P' = \mathbb{Z}_p^2$.

About meta-abelian p-groups:

1. $P' = \mathbb{Z}_p$, **extra-special** p**-group**

2. $P' = \mathbb{Z}_p^k$,

Sergeicuk, V. V. The classification of metabelian p -groups. (Russian) Matrix problems (Russian), pp. 150-161. Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1977.

Visneveckii, A. L., Groups class 2 and exponent p **with commutator group** $\mathbb{Z}_p^2$**, Doll, Akad. Nauk Ukrain. SSR Ser, 1980, No 9, 9-11. 1980.**

Scharlau, Rudolf, Paare alternierender Formen. Math. Z. 147 (1976), no. 1, 13-19.

$\widetilde{G}/\mathbb{Z}_p^2 = (\mathbb{Z}_p^s \times \mathbb{Z}_p^s) \rtimes H$, $H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p)$, $\widetilde{G} = P.\widetilde{H}$

Transitive subgroups H_1 of $\mathrm{GL}(s,p)$:

$\mathrm{SL}(d,q) \leq H_1 \leq P\Gamma L(d,q)$, $q^d = p^s$
$Sp(d,q) \lhd H_1$, q^{2d}
$G_2(q) \lhd H_1$, q^6
$SL(2,3) \lhd H_1$, $q = 5^2, 7^2, 11^2, 23^2$
A_6, 2^4
A_7, 2^4
$PSU(3,3)$, 2^6
$SL(2,13)$, 3^3.

1. Huppert, Bertram Zweifach transitive, auflsbare Permutationsgruppen. (German) Math. Z. 68 1957 126-150.

2. Hering, Christoph, Transitive linear groups and linear groups which contain irreducible subgroups of prime order. Geometriae Dedicata 2 (1974), 425-460.

3. Hering, Christoph Zweifach transitive Permutationsgruppen, in denen 2 die maximale Anzahl von Fixpunkten von Involutionen ist. (German) Math. Z. 104 1968 150-174.

What can we do without classification of P and $P.\tilde{H}$?

Try to do that by combining group theoretical approached and lifting theorem !

$$\widetilde{T}/K = T = \mathbb{Z}_p^s \times \mathbb{Z}_p^s,$$

$$\widetilde{T} = \langle \widetilde{T}_{\widetilde{w}}, \widetilde{T}_{\widetilde{u}} \rangle = (K \times \widetilde{T}_{\widetilde{w}}) \rtimes \widetilde{T}_{\widetilde{u}},$$

$$K = \langle z_1, z_2 \rangle = \widetilde{T}' = Z(\widetilde{T}) \cong \mathbb{Z}_p^2,$$

$$L := \widetilde{T}_{\widetilde{w}} = \langle a_i \mid 1 \le i \le s \rangle, \quad R := \widetilde{T}_{\widetilde{u}} = \langle b_i \mid 1 \le j \le s \rangle,$$

$$[a_i, b_j] = z_1^{\alpha_{ij}} z_2^{\beta_{ij}}, \quad \alpha_{ij}, \beta_{ij} \in \mathbb{F}_p,$$

$$\mathrm{A} := (\alpha_{ij})_{s\times s} \quad \text{and} \quad \mathrm{B} := (\beta_{ij})_{s\times s}.$$

For any $\ell = \prod_{i=1}^s a_i^{\alpha_i} \in L$ **and** $r = \prod_{i=1}^s b_i^{\beta_i} \in R$**,**

$$[\ell, r] = z_1^{\alpha \mathrm{A} \beta^{\mathbf{T}}} z_2^{\alpha \mathrm{B} \beta^{\mathbf{T}}}, \ \alpha = (\alpha_1, \cdots, \alpha_s), \ \beta = (\beta_1, \cdots, \beta_s).$$

Our approaches:

Step 1. Get all coset graphs from $\widetilde{T}$ (easy to do)

Step 2. Find the voltage graphs (covering graphs) form the above coset graphs (with nice voltage assignment)

Step. Show the above coving graphs satisfy the 2-ar-transitivity.

Theorem

We may take $\mathrm{A} = \mathrm{I}$ *and* $\mathrm{B} = \mathrm{M}$,

$$\mathrm{M_d} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -a_{d-1} \end{pmatrix}_{d \times d}$$

$$\mathrm{M} = \begin{pmatrix} \mathrm{M_d} & 0 & 0 & \dots & 0 \\ 0 & \mathrm{M_d} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{M_d} \end{pmatrix}_{s \times s},$$

where $d \geq 2$, $d \mid s$ *and* $\varphi(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ *is an irreducible polynomial of degree* d *over* $\mathbb{F}_p$.

$$X \cong X(s, p, \varphi(x)) = Y \times_f K : f_{\alpha,\beta'} = (\beta\alpha^T, \beta\mathrm{M}\alpha^T),$$

Proof

Step 1: Show that $|\mathrm{A}|, |\mathrm{B}| \neq 0$.

Consider the quotient graph induced by $\langle z_1^i z_2^j \rangle$ of order p, which is a p-fold cover of K_{p^s,p^s}.

Then $\widetilde{T}/\langle z_1^i z_2^j \rangle$ is an extraspecial p-group and $Z(\widetilde{T}/\langle z_1^i z_2^j \rangle)$ is of order p.

Take $i = 1$ and $j = 0$. In $\widetilde{T}/\langle z_1 \rangle$, we have $[\bar{\ell}, \bar{r}] = \bar{z}_2{}^{\alpha \mathrm{B} \beta^{\mathbf{T}}}$

If $|\mathrm{B}| = 0$, them take $\beta_1 \neq 0$ such that $\mathrm{B}\beta_1^{\mathbf{T}} = 0$, which implies $\alpha \mathrm{B} \beta_1^{\mathbf{T}} = 0$ for any α. Therefore, for the corresponding element r_1, we have $[\bar{\ell}, \bar{r}_1] = \bar{1}$ for any ℓ.

Now, $\bar{r}_1 \in Z(\widetilde{T}/\langle z_1 \rangle) \setminus (K/\langle z_1 \rangle)$ and so $\langle \bar{z}_2, \bar{r}_1 \rangle \leq Z(\widetilde{T}/\langle z_1 \rangle)$ is of order at least p^2, a contradiction.

Hence, $|\mathrm{B}| \neq 0$. Similarly, $|\mathrm{A}| \neq 0$.

Step 2: Show that $\mathrm{A} = \mathrm{I}$.

For $\mathrm{P} = (p_{ij})_{s\times s}$, $\mathrm{Q} = (q_{ij})_{s\times s} \in \mathrm{GL}(s,p)$, **set**
$a'_i = \prod_{\ell=1}^{s} a_\ell^{p_{\ell i}} \quad \text{and} \quad b'_j = \prod_{\ell=1}^{s} b_\ell^{q_{\ell j}}$.
$[a'_i, b'_j] = z_1^{\alpha'_{ij}} z_2^{\beta'_{ij}}$,

where $(\alpha'_{ij})_{s\times s} = \mathrm{P}^{\mathbf{T}}\mathrm{A}\mathrm{Q}, \quad (\beta'_{ij})_{s\times s} = \mathrm{P}^{\mathbf{T}}\mathrm{B}\mathrm{Q}$.
Take $\mathrm{P} = (\mathrm{A}^{-1})^{\mathbf{T}}$ **and** $\mathrm{Q} = \mathrm{I}$. **Then we get** $(\alpha'_{ij})_{s\times s} = \mathrm{I}$.
Hence, assume

$$[\ell, r] = z_1^{\alpha\beta^{\mathbf{T}}} z_2^{\alpha\mathrm{B}\beta^{\mathbf{T}}}.$$

Step 3: Find the conditions for the matrix B.

Recall H **lifts to** $\widetilde{H}$ **and** $\widetilde{G} = ((K \times L) \rtimes R)\widetilde{H}$**. Then for any** $\widetilde{h} \in \widetilde{H}$, **set**

$$a_i^{\widetilde{h}} = (\Pi_{j=1}^{s} a_j^{p_{ji}})k_{i1}, \quad b_i^{\widetilde{h}} = (\Pi_{j=1}^{s} b_j^{q_{ji}})k_{i2}, \quad z_1^{\widetilde{h}} = z_1^a z_2^b, \quad z_2^{\widetilde{h}} = z_1^c z_2^d, \tag{1}$$

where $i = 1, 2, \cdots s$**,** $k_{i1}, k_{i2} \in K$ **and moreover, set**

$$\mathrm{P} = (p_{ij})_{s\times s}, \quad \mathrm{Q} = (q_{ij})_{s\times s} \in \mathrm{GL}(s, p).$$

Since $[\ell, r] = z_1^{\alpha\beta^{\mathbf{T}}} z_2^{\alpha\mathrm{B}\beta^{\mathbf{T}}}$, **we have**

$$[\ell^{\widetilde{h}}, r^{\widetilde{h}}] = z_1^{\alpha\mathrm{P}^{\mathbf{T}}\mathrm{Q}\beta^{\mathbf{T}}} z_2^{\alpha\mathrm{P}^{\mathbf{T}}\mathrm{B}\mathrm{Q}\beta^{\mathbf{T}}} = z_1^{a\alpha\beta^{\mathbf{T}}+c\alpha\mathrm{B}\beta^{\mathbf{T}}} z_2^{b\alpha\beta^{\mathbf{T}}+d\alpha\mathrm{B}\beta^{\mathbf{T}}},$$

which forces that

$$\mathrm{P}^{\mathbf{T}}\mathrm{Q} = a\mathrm{I} + c\mathrm{B}, \quad \mathrm{P}^{\mathbf{T}}\mathrm{B}\mathrm{Q} = b\mathrm{I} + d\mathrm{B}.$$

Then we have

$$(a\mathrm{I} + c\mathrm{B})\mathrm{Q}^{-1}\mathrm{B}\mathrm{Q} = (b\mathrm{I} + d\mathrm{B}). \tag{2}$$

$\varepsilon : \widetilde{h} \to \mathrm{Q}$ **gives an homomorphism from** $\widetilde{H}$ **to** $\mathcal{H} := \varepsilon(\widetilde{H})$. **Then** $\mathcal{H}$ **acts transitively on** $V \backslash \{0\}$.

Let $S = \{f(\mathrm{B}) \mid f(x) \in \mathbb{F}_p[x]\}$, **a subalgebra of** $\mathrm{Hom}_{\mathbb{F}_p}(V, V)$.

Let $S^* = \{f(\mathrm{B}) \in S \mid |f(\mathrm{B})| \neq 0\} \subset S$.

Then S^* **forms a group of** $\mathrm{GL}(s, p)$ **(finiteness of** S**) .**

Since $\mathrm{P}^{\mathbf{T}}\mathrm{Q} = a\mathrm{I} + c\mathrm{B} \in S^*$, **we have** $(a\mathrm{I} + c\mathrm{B})^{-1}$ **is contained in** S^*.

$$\mathrm{Q}^{-1}\mathrm{B}\mathrm{Q} = (a\mathrm{I} + c\mathrm{B})^{-1}(b\mathrm{I} + d\mathrm{B}) \in S^*.$$

That is, $\mathcal{H}$ **normalizes** S.

Step 4: Show S is a field.

Consider S-right module V. For any $v \in V$, vS is irreducible.

In fact, let V_1 be an irreducible S-submodule of vS.
Take $g \in \mathcal{H}$ such that $vg \in V_1$. Then
$\dim(V_1) \leq \dim(vS) = \dim(vSg) = \dim(vgS) \leq \dim(V_1S) = \dim(V_1)$.
Hence, $\dim(V_1) = \dim(vS)$**, that is $vS = V_1$.**

Take any $s \in S \setminus S^*$. Then $vs = 0$ for some $v \in V \setminus \{0\}$ and so $(vS)s = vsS = 0$. **For any $w \in V \setminus vS$, we have $vS \neq wS$ If** $wS = (v + w)S$, **then $v \in wS$ forcing $wS = vS$, a contradiction. Therefore,** $wS \neq (v + w)S$, **which means** $wS \cap (v + w)S = \{0\}$**. Since $vs = 0$, we have** $= vs + ws = (v + w)s \in wS \cap (v + w)S = \{0\}$**. By the arbitrary of $w \in V \setminus vS$ and $(vL)s = 0$, we get $us = 0$ for any vector $u \in V$ and so $s = 0$.**

Therefore, S is a field

Step 5: Determination of B.

Let $p(x) = \sum_{i=0}^{d} a_i x^i$ be the minimal monic polynomial for B. Since $S = \mathbb{F}_p(B)$ is a field, $p(x)$ is irreducible, and $\mathrm{I}, \mathrm{B}, \mathrm{B}^2, \cdots \mathrm{B}^{d-1}$ is a base of S over $\mathbb{F}_p$.

Set $V = \bigoplus_i v_i S$, where every $v_i S$ is an irreducible S-module of dimension d. Clearly, $d \Big| s$ so that $1 \le i \le \frac{s}{d}$.

Define $\mathcal{B}(v) = v\mathrm{B}$ for any $v \in V$. Then $(e_1, \cdots, e_s)\mathcal{B} = (e_1, \cdots e_s)\mathrm{B}^{\mathbf{T}}$, $e_1, \cdots e_s$ are unit vectors.

V has a base:

$$v_1, v_1\mathrm{B}, \cdots, v_1\mathrm{B}^{d-1}; v_2, v_2\mathrm{B}, \cdots, v_2\mathrm{B}^{d-1}; \cdots; v_{\frac{s}{d}}, v_{\frac{s}{d}}\mathrm{B}, \cdots, v_{\frac{s}{d}}\mathrm{B}^{d-1}.$$

Under this base, the matrix of $\mathcal{B}$ is exactly M. Therefore, $\mathrm{B} \sim \mathrm{B}^{\mathbf{T}} \sim \mathrm{M}$, and we may let $\mathrm{B} = \mathrm{M}$.

Step 6: Show X is isomorphic to $X(s,p,\varphi(x)) = Y \times_f K$, $f_{\alpha,\beta'} = (\beta\alpha^{\mathbf{T}}, \beta\mathrm{M}\alpha^{\mathbf{T}})$.

$X \cong X_1 := \mathbf{B}(\widetilde{T}, L, R; RL)$, **recall** $\widetilde{T} = (K \times L) \rtimes R$.

Connectedness and valency: $\langle (LR)(LR)^{-1} \rangle = \langle L, R \rangle = \widetilde{T}$ **and** $|RL : L| = p^s$

Cover: the quotient graph $\overline{X}_1$ **induced by the center** K **is** K_{p^s,p^s}.

For any $\ell = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_s^{\alpha_s} \in L$ **and** $r = b_1^{\beta_1} b_2^{\beta_2} \cdots b_s^{\beta_s} \in R$, **define** $\phi(\ell) = (\alpha_i) \quad \text{and} \quad \phi(r) = (\beta_i)$.
L **is adjacent to** $\{R\ell \mid \ell \in L\}$; Lr **is adjacent to**

$$\{R\ell[\ell, r] \mid \ell \in L\} = \{R\ell z_1^{\phi(\ell)\phi(r)^{\mathbf{T}}} z_2^{\phi(\ell)\mathrm{M}\phi(r)^{\mathbf{T}}} \mid \ell \in L\}.$$

Then $X_1 \cong X(s,p,\varphi(x))$ **by the map** ψ**:**

$$\psi(Lrz_1^i z_2^j) \;=\; (\phi(r),(i,j)), \quad \psi(R\ell z_1^i z_2^j) \;=\; (\phi(\ell)',(i,j)),$$

where $r \in R$, $\ell \in L$ **and** $z_1^i z_2^j \in K$.

Step 7: Show that for $X(s, p, \varphi(x))$, its fibre-preserving automorphism group acts 2-arc-transitively.

For $Y = K_{p^s,p^s}$, let $T_1 \cong T_2 \cong \mathbb{Z}_p^s$ such that T_1 (resp. T_2) translates the vectors in U (resp. W) and fixes W (resp. U) pointwise.

(i) Clearly, for the graph $X(s, p, \varphi(x))$, both T_1 and T_2 lifts.

(ii) V **is a space over** $S = \mathbb{F}_p(\mathrm{M})$**, where** $\mathrm{M} = \mathrm{B}$**.**

Let $\mathcal{C}$ **be the centralizer of** S^* **in** $\mathrm{GL}(s,p)$**. Then** $S^* \le \mathcal{C}$ **and for any** $c \in \mathcal{C}$**,** $\ell \in S$ **and** $v \in V$**, we have** $(vs)c = (vc)s$**, that is** c **induces a linear transformation on the** S**-space** V**. Therefore,** $\mathcal{C} \le \mathrm{GL}(V,S) \cong \mathrm{GL}(\frac{s}{d}, S)$. **In particular,** $\mathcal{C}$ **is transitive on** V **(**$\mathcal{C}$ **contains a Single-subgroup).**

For any $\mathrm{P} \in \mathcal{C}$**, define a map** ρ_{P} **on** $V(Y)$ **by**

$$\alpha^{\rho_{\mathrm{P}}} = \alpha \mathrm{P}^{\tau} \quad \text{and} \quad (\alpha')^{\rho_{\mathrm{P}}} = (\alpha \mathrm{P})'$$

for any $\alpha \in V(s,p)$**, where** τ **denotes the inverse transpose automorphism of** $\mathrm{GL}(s,p)$**. Set**

$$H := \langle \rho_{\mathrm{P}} \mid \mathrm{P} \in \mathcal{C} \rangle \le \mathbf{Aut}\,(Y).$$

Then $H \cong \mathcal{C}$ **and** H **acts transitively on nonzero vectors on both biparts of** Y**.**

For any $\rho_{\mathrm{P}} \in H$, we have

$$\begin{aligned} f_{\alpha^{\rho_{\mathrm{P}}},(\beta')^{\rho_{\mathrm{P}}}} &= f_{\alpha \mathrm{P}^{\tau},(\beta \mathrm{P})'} = (\beta \mathrm{P}(\alpha \mathrm{P}^{\tau})^{\mathbf{T}}, \beta \mathrm{PM}(\alpha \mathrm{P}^{\tau})^{\mathbf{T}}) \\ &= (\beta \alpha^{\mathbf{T}}, \beta \mathrm{PMP}^{-1} \alpha^{\mathbf{T}}) = (\beta \alpha^{\mathbf{T}}, \beta \mathrm{M} \alpha^{\mathbf{T}}) = f_{\alpha,\beta'}. \end{aligned}$$

Thus, we get $f_{W^{\rho_{\mathrm{P}}}} = f_W$ for any closed walk W in Y. By Lifting Theorem, ρ_{P} lifts and so H lifts.

(iii) Take a matrix Q such that $\mathrm{QMQ}^{-1} = \mathrm{M}^{\mathbf{T}}$. Define $\sigma \in$ Aut (Y): $\alpha^{\sigma} = (\alpha\mathrm{Q})'$ and $\beta'^{\sigma} = \beta\mathrm{Q}^{\tau}$ for any $\alpha, \beta \in V(s,p)$. Then

$$\begin{aligned} f_{\alpha^{\sigma},(\beta')^{\sigma}} &= f_{(\alpha\mathrm{Q})',\beta\mathrm{Q}^{\tau}} = -f_{\beta\mathrm{Q}^{\tau},(\alpha\mathrm{Q})'} \\ &= -(\alpha\mathrm{Q}(\beta\mathrm{Q}^{\tau})^{\mathbf{T}}, \alpha\mathrm{QM}(\beta\mathrm{Q}^{\tau})^{\mathbf{T}}) \\ &= -(\alpha\beta^{\mathbf{T}}, \alpha\mathrm{QMQ}^{-1}\beta^{\mathbf{T}}) = -(\beta\alpha^{\mathbf{T}}, \alpha\mathrm{M}^{\mathbf{T}}\beta^{\mathbf{T}}) \\ &= -(\beta\alpha^{\mathbf{T}}, \beta\mathrm{M}\alpha^{\mathbf{T}}) = -f_{\alpha,\beta'}. \end{aligned}$$

Thus, $f_{W^{\sigma}} = -f_W$ for any closed walk W. So σ lifts.

(iv) Check:

$$(t_\alpha)_1^\sigma = (t_{\alpha\mathrm{Q}})_2 \in T_2, \quad (t_\beta)_2^\sigma = (t_{\beta\mathrm{Q}^\tau})_1 \in T_1, \quad (\rho_{\mathrm{P}})^\sigma = \rho_{\mathrm{Q}^{-1}\mathrm{P}^\tau\mathrm{Q}}.$$

Set

$$A := ((T_1 \times T_2) \rtimes H)\langle\sigma\rangle \leq \mathbf{Aut}\,(Y).$$

Then, A acts 2-arc-transitively on Y. By (i)-(iii), we know that A lifts so that the fibre-preserving automorphism group of the graph $X(s, p, \varphi(x))$ acts 2-arc-transitively. □

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End

Thank You Very Much !