8th PhD Summer School in Discrete Maths
Finite Permutation Groups
Lecture 2: Imprimitivity and decompositions

Colva M. Roney-Dougal
colva.roney-dougal@st-andrews.ac.uk

Rogla, 3 July 2018

University of St Andrews

Equivalence of representations

Definition 17

Let $\rho : G \to \mathrm{Sym}(\Omega)$, $\tau : G \to \mathrm{Sym}(\Gamma)$ be perm reps. Then ρ and τ are equivalent if there exists a bijection $\lambda : \Omega \to \Gamma$ s.t. for all $\omega \in \Omega$ and $g \in G$:

$$(\omega^{(g\rho)})\lambda = (\omega\lambda)^{(g\tau)}.$$

Actions are equivalent iff the corresponding perm reps are.

Lemma 18

Let G act transitively on Ω and Γ, and let $H = G_\omega$ for some $\omega \in \Omega$. The actions are equivalent iff $H = G_\gamma$ for some $\gamma \in \Gamma$.

Theorem 19

Let G act transitively on Ω, with point stabiliser H. This action is equivalent to the action μ_H on the right cosets of H.
Two right coset actions μ_H and μ_K are equivalent iff H and K are conjugate subgroups of G.

§4: Semidirect and wreath products

Automorphism groups

Defn: An automorphism of a gp G is an isomorphism $\phi : G \to G$.

Defn: $\mathrm{Aut}(G) = \{\phi \;:\; \phi \text{ is an automorphism of } G\} \subseteq \mathrm{Sym}(G)$ is the automorphism group of G.

Lemma 20

$\mathrm{Aut}(G) \leq \mathrm{Sym}(G)$.

Defn: Let $g \in G$. The map $c_g : G \to G$, $x \mapsto g^{-1}xg$ is an inner automorphism of G.

$\mathrm{Inn}(G) = \{c_g \;:\; g \in G\} \subseteq \mathrm{Aut}(G)$ is the inner automorphism group of G.

Theorem 21

$\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

Defn: $Z(G) = \{g \in G \;:\; gh = hg \;\forall\; h \in G\}$ is the centre of G.

Theorem 22

$\mathrm{Inn}(G) \cong G/Z(G)$.

Extensions and semi-direct products

Defn: G – group, $1 < N \lhd G$. If $G/N \cong H$ then G is an extension of N by H. Write $G = N.H$.

Defn: N, H – groups. $\phi : H \to \mathrm{Aut}(N)$ homom. The (external) semi-direct product of N by H w.r.t. ϕ is

$$N :_\phi H = \{(n, h) \ : \ n \in N, h \in H\}$$

with product

$$(n_1, h_1)(n_2, h_2) = (n_1(n_2^{(h_1^{-1}\phi)}), h_1 h_2).$$

Often just write $N : H$ if the action of N on H is clear.

Theorem 23

Let N, H and ϕ be as above. Then

1. *$G = N :_\phi H$ is a group.*
2. *G has a normal subgroup $\overline{N} \cong N$ and a subgroup $\overline{H} \cong H$, s.t. $\overline{N} \cap \overline{H} = 1$, and $G/\overline{N} \cong \overline{H}$.*

Internal semi-direct products

Defn: G – group s.t. $\exists\ 1 \leq N \trianglelefteq G$ and $1 \leq H \leq G$ s.t.

1. $N \cap H = 1$
2. $NH = G$.

Then G is an (internal) semi-direct product or split extension of N by H. Write $G = N : H$.

Notice: $n_1 h_1 \cdot n_2 h_2 = n_1 h_1 n_2 h_1^{-1} h_1 h_2 = n_1 n_2^{h_1^{-1}} h_1 h_2$.

Theorem 24

Let G be an internal semi-direct product of N by H. Then H acts on N via automorphisms. Let $\phi : H \mapsto \mathrm{Aut}(N)$ be the corresponding perm rep. Then $G \cong N :_\phi H$.

Wreath products

Defn: Let K be a group, and let H act on $\underline{n}$. The wreath product of K by H, written $K \wr H$, is the semidirect product $K^n : H$, where H acts on K^n by permuting co-ordinates:

$$(k_1, \dots, k_n)^h = (k_{1^{h^{-1}}}, \dots, k_{n^{h^{-1}}}).$$

Why is the action "backwards"?
Write (k_1, k_2, k_3) as $\{(k_1, 1), (k_2, 2), (k_3, 3)\}$. Let $\sigma = (1\ 2\ 3)$ act on the second co-ordinates: $\{(k_1, 2), (k_2, 3), (k_3, 1)\}$.
Which k_i now goes with second entry 1? $k_{1^{\sigma^{-1}}}$.

Product in $K \wr H$ is:
$(k_{11}, \dots, k_{1n})h_1(k_{21}, \dots, k_{2n})h_2 = (k_{11}k_{21^{h_1}}, \dots, k_{1n}k_{2n^{h_1}})h_1h_2.$

Theorem 25
$K \wr H$ is a group of order $|K|^n|H|$.

The base group is $K^n = \{(k_1, \dots, k_n)1_H \ : \ k_i \in K\} \trianglelefteq K \wr H$.

§5: Imprimitivity

Blocks

Defn: Let $\Delta \subseteq \Omega$ be a nonempty subset of Ω, and let G act transitively on Ω. Then Δ is a block for G if for all $g \in G$ either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$.

Defn: Let G act transitively on Ω. If all blocks have size 1 or $|\Omega|$ then G is primitive, otherwise G is imprimitive.

Lemma 26

Let G act transitively on Ω, and let $\Delta \subseteq \Omega$ be a block for G. Then

$$\Sigma = \{\Delta^g \ : \ g \in G\}$$

is a partition of Ω, and each set in Σ is a block for G.

Σ is a system of imprimitivity for G.

If G is imprimitive then G acts on Σ, as well as on Ω.

Point stabilisers in primitive groups

Defn: Let G act on Ω, and let $\Delta \subset \Omega$. The setwise stabiliser of Δ in G is

$$G_{\{\Delta\}} = \{g \in G \ : \ \Delta^g = \Delta\}.$$

Lemma 27

1. *$G_{\{\Delta\}} \leq G$.*
2. *If G is transitive and Δ is a block for G, then $G_{\{\Delta\}}$ is transitive on Δ.*

Theorem 28

Let G act transitively on Ω, with $|\Omega| \geq 2$. Let $\alpha \in G$.
G is primitive if and only if G_α is a maximal subgroup of G.

Corollary 29

A regular permutation group is primitive iff it has prime degree.

The imprimitive action of a wreath product

Lemma 30

Let K act on Δ, and let H act on $\underline{n}$. Then $G = K \wr H$ acts on $\Delta \times \underline{n}$ by

$$(\delta, i)^{(k_1,\dots,k_n)h} = (\delta^{k_i}, i^h).$$

Lemma 31

If $n > 1$ and K and H are transitive, then the action of $K \wr H$ in Lemma 30 is transitive but imprimitive, with blocks

$$\Delta \times \{i\} = \{(\delta, i) \ : \ \delta \in \Delta\}$$

for each $i \in \underline{n}$.

§6: Decompositions of permutation groups

Intransitive to transitive

Lemma 32
Let $G \leq \mathrm{Sym}(\Omega)$ be intransitive, and let $\Delta = \alpha^G$. Then the map $G \to \mathrm{Sym}(\Delta)$ that sends each $g \in G$ to the permutation of Δ that it induces is a homomorphism.

Defn: This map is the restriction of G to Δ. Write as $G \to G^\Delta$, $g \mapsto g|_\Delta$.
The image $G^\Delta \leq \mathrm{Sym}(\Delta)$ is a transitive constituent of G.

Abuse notation: think of G^Δ as a subgp of $\mathrm{Sym}(\Omega)$, fixing $\Omega \setminus \Delta$.

Defn: Let $H \leq G_1 \times \cdots \times G_k$. Then H is a subdirect product of $G_1, \ldots, G_k$ if $\forall i \in \underline{k}$, $\forall g_i \in G_i$, $\exists\ h = (h_1, \ldots, h_k) \in H$ s.t. $h_i = g_i$.

Theorem 33
Let $G \leq \mathrm{Sym}(\Omega)$, and let $\Delta_1, \ldots, \Delta_k$ be the orbits of G. Then G is equal to a subdirect product of $G^{\Delta_1} \times \cdots \times G^{\Delta_k}$.