Vertex- and edge-transitivity in products of finite and infinite graphs, and in graphs of polynomial growth

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Preface

This talk contains a collection of results related to the courses of Colva Roney-Dougal on finite permutation groups and that of Gabriel Verret on vertex-transitive graphs and their local actions.

The first part pertains to s-transitive infinite graphs, supplementing results about finite s-transitive graphs presented by Gabriel Verret.

The second part treats edge- and vertex-transitivity of lexicographic, Cartesian, direct and strong products of finite graphs and edge- and vertex-transitivity of weak Cartesian products of finite and infinite graphs. It also illustrates results about products of permutation groups presented by Colva Roney-Dougal.

The third is about graphs with primitive automorphisms groups and the role of powers of the Johnson graph with respect to the Cartesian product in the investigation of graphs with primitive automorphism groups.

We end with the Infinite Motion Conjecture of Tom Tucker.

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1. s-Transitivity and Growth in Graphs

In this part is concerned with the interplay of s-transitivity, growth, and the structure of infinite graphs and their groups.

It mainly lists some of the results that were collected in a 1991 survey on graphs with polynomial growth*.

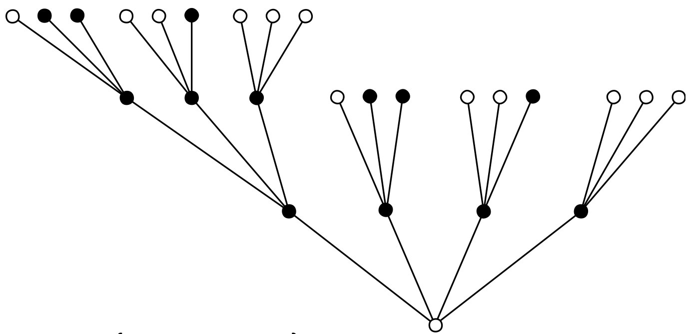
Several basic definitions from Gabriel Verret's course, such as that of s-transitivity or primitivity, are omitted.

*Imrich and Seifter, A survey on graphs with polynomial growth, Discrete Mathematics 95 (1991), 101 – 117.

Let Γ be the class of connected, locally finite* infinite graphs.

For a graph $G \in \Gamma$ the ball of radius n and center $v_0 \in V(G)$ is
$$B_n(v_0) = \{v \in V(G) : d(v_0, v) \leq n\}.$$

For the homogenous tree T_d of degree $d > 2$ we have
$$|B_n(v_0)| \geq (d-1)^n.$$



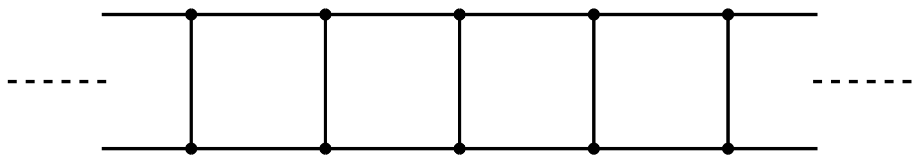
*Every vertex has finite degree (valence).

We say T_d has exponential growth $(d-1)^n$.

If $\exists k$ such that $|B_n(v_0)| = O(n^k)$, then G has polynomial growth*.

It is clear what we mean by linear and quadratic growth.

For example, the two-sided infinite path, rays and the ladder depicted below have linear growth.



Note that we defined growth only for graphs in Γ.

*k is independent of the base point v_0.

Theorem (Godsil, Imrich, Seifter, Watkins, Woess 1989*) *Let G be a vertex-transitive graph of polynomial growth. Then G is imprimitive.*

We have the following generalization of a result of Weiss 1981:

Theorem (Seifter 1991†) *Let G be an s-transitive graph of polynomial growth and valence at least 3 in Γ. Then $s \leq 7$.*

Compare this with the fact that T_d, which has exponential growth for $d > 2$, is s-transitive for any $s \geq 0$.

*A note on bounded automorphisms of infinite graphs, Graphs and Combinatorics 5 (1989), 339 – 349)

†Properties of graphs with polynomial growth, J. Combin. Theory Ser. B 52 (1991) 222–235.

An automorphism α is bounded if there is a constant k depending on α such that $d(v, \alpha(v)) \leq k$ for all $v \in V(G)$. The bounded automorphisms form a group $B(G)$.

Theorem (Seifter 1991*) *Let G be a graph of valence at least 3 in Γ. If $B(G)$ is transitive on $V(G)$, then G cannot be 3-transitive.*

Theorem (Seifter 1991) *Let G be an s-transitive graph with polynomial growth in Γ. If G has valence at least 3 and $s \geq 2$, then* Aut(G) *is a finitely generated almost nilpotent group.*

*Both results on this page are from Seifter, Properties of graphs with polynomial growth, J. Combin. Theory Ser. B 52 (1991) 222–235.

Let G_1, G_2 be graphs. A homomorphism $p : V(G_1) \to V(G_2)$ of $V(G_1)$ onto $V(G_2)$ is a covering, if it is a local isomorphism.

In other words, for every $v \in V(G_1)$ the mapping p induces a bijection between the edges incident with v and those incident with $p(v)$.

Theorem (Godsil, Seifter 1991*) *Let G be an s-transitive graph, $s \geq 0$, of polynomial growth in Γ. Then there exist infinitely many finite graphs $H_1, H_2, \ldots$ such that:*

(1) G is a covering graph of every H_i, $i \geq 1$.

(2) Each H_k is a covering graph of $H_1, \ldots H_{k-1}$.

(3) If $s \geq 2$, then each H_i is at least s-transitive.

*Graphs with polynomial growth are covering graphs, Graphs and Combinatorics 8 (1992), 233-241

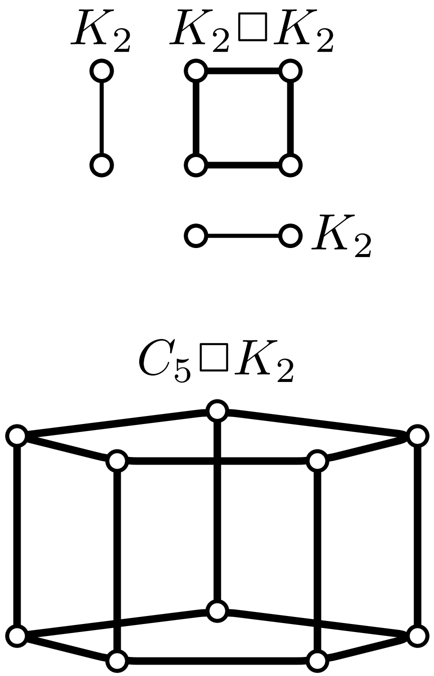
2. Products of graphs

The Cartesian product $G \,\square\, H$, the direct product $G \times H$, the strong product $G \boxtimes H$, and the lexicographic product $G \circ H$, each has as its vertex set the Cartesian product $V(G) \times V(H)$.

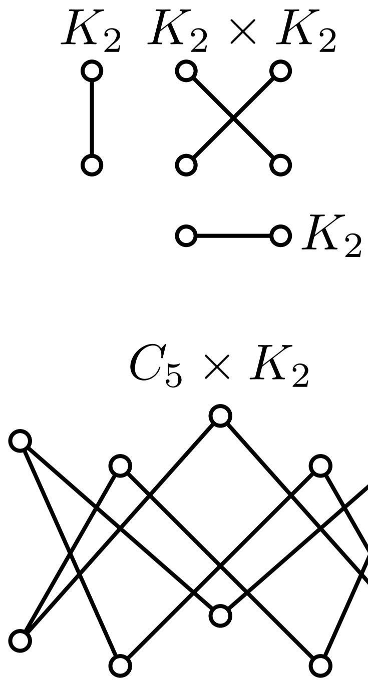
Edges are as follows:

$$\begin{aligned}
E(G \,\square\, H) &= \Big\{(x,u)(y,v) \mid (xy \in E(G) \wedge u = v) \vee (x = y \ \wedge \ uv \in E(H))\Big\}, \\
E(G \times H) &= \Big\{(x,u)(y,v) \mid xy \in E(G) \ \wedge \ uv \in E(H)\Big\}, \\
E(G \boxtimes H) &= E(G \,\square\, H) \cup E(G \times H), \\
E(G \circ H) &= \Big\{(x,u)(y,v) \mid xy \in E(G) \vee (x = y \ \wedge \ uv \in E(H))\Big\}.
\end{aligned}$$

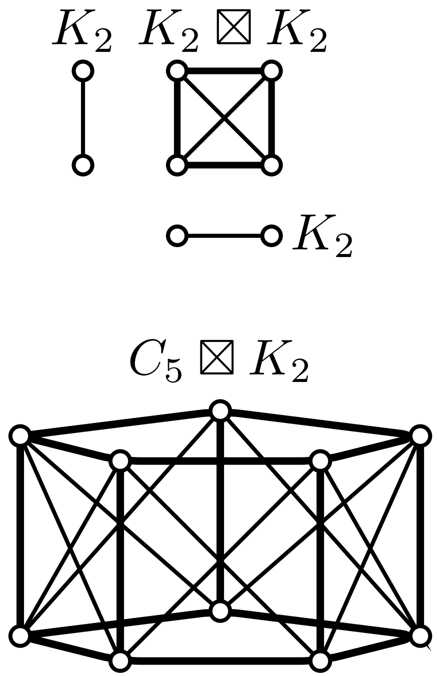
Examples



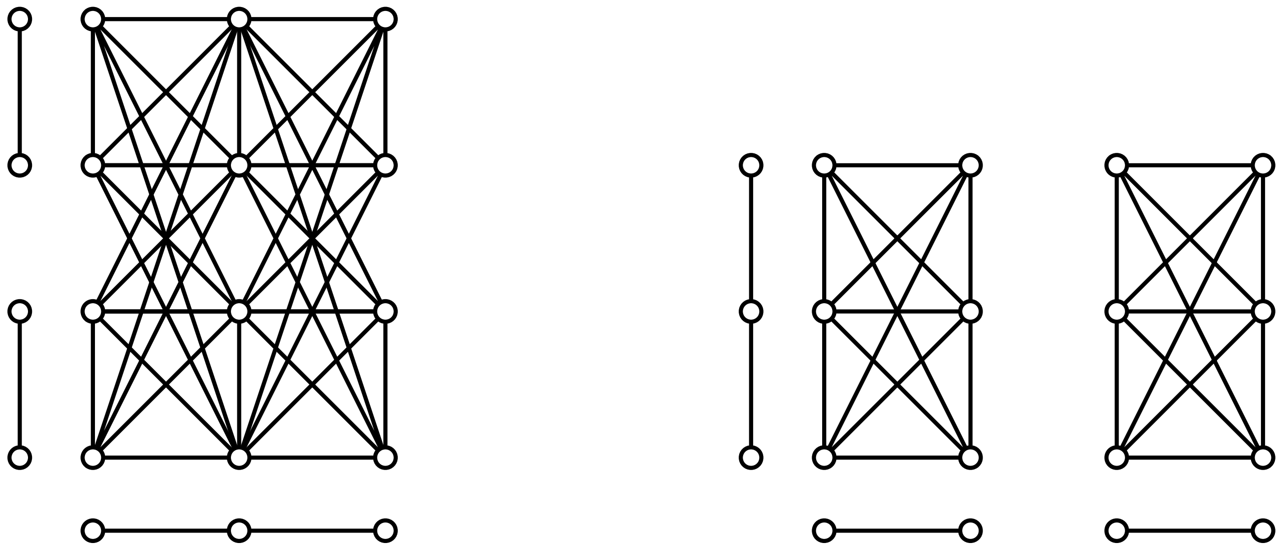
Cartesian product



Direct Product



Strong product



Lexicographic product

This is the only non-commutative product of the four products treated here.

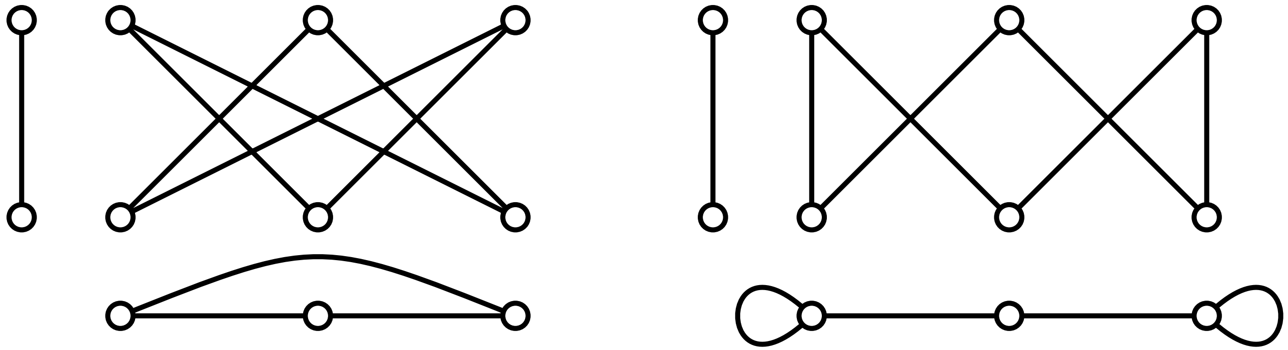
2.a. Vertex transitivity in finite products

The Cartesian, the strong and lexicographic product of two finite graphs are vertex transitive if and only if the factors are.

For the direct product this is true if the factors are non-bipartite.

For proofs we refer to the Handbook of Product Graphs by Hammack, Imrich and Klavžar, CRC Press 2011.

In the case of bipartite graphs products can be vertex transitive even when not all factors are vertex-transitive:



Note that we admit loops for the direct product in order to have a unit. It is the one vertex graph with a loop, denoted K_1^*.

(We will also write K_n^* for the complete graph K_n with a loop at every vertex.)

2.b. Edge-transitivity in lexicographic products of finite graphs*

Theorem *Let G be a connected graph that is not complete and H be any graph. Then $G \circ H$ is edge-transitive if and only if G is edge-transitive and H is edgeless.*

Theorem *The lexicographic product $G \circ H$ of a non-trivial complete graph G by a graph H is edge-transitive if and only if H is the product of a complete graph by an edgeless graph.*

This means that $G \circ H$ can be represented in the form $K \circ N$, where K is complete and N edgeless.

*W. Imrich, A. Iranmanesh, S. Klavžar and A. Soltani, Edge-transitive lexicographic and Cartesian products, Discuss. Math. Graph Theory 36 (2016), 857–865.

2.c. Edge-transitivity in Cartesian products of finite graphs*

Theorem *A connected graph that is not prime with respect to the Cartesian product is edge-transitive if and only if it is the power of a connected, edge- and vertex-transitive graph.*

Corollary *A connected graph that is not prime with respect to the Cartesian product G is half-transitive if and only if it is the power of a connected, half-transitive graph.*

*W. Imrich, A. Iranmanesh, S. Klavžar and A. Soltani, Edge-transitive lexicographic and Cartesian products, Discuss. Math. Graph Theory 36 (2016), 857–865.

2.d. Edge-transitivity in non-bipartite finite direct products*

Theorem *Suppose $A \times B$ is connected and non-bipartite. Then it is edge-transitive if and only if either*

(1) both factors are edge-transitive and at least one is arc-transitive,

(2) or one factor is edge-transitive (and non-trivial) and the other is a K_n^*.

*Unless otherwise stated, the results in Sections 2.d - 2.f are from Hammack, Imrich and Klavžar, Edge-transitive products, J. Algebraic Combin. 43 (2016), 837–850.

Corollary *Every connected, non-trivial, edge-transitive non-bipartite graph G has form $G = K_n^* \times H$ (possibly with $n = 1$), where*

H is non-trivial, has no factor K_n^*, and at most one half-transitive (prime) factor, while all other (prime) factors, if any, are arc-transitive.

Furthermore, G is half-transitive if H has a half-transitive factor. Otherwise G is arc-transitive.

For bipartite graphs we have the following proposition:

Proposition *Suppose A has an odd cycle and B is bipartite. If both $A \times K_2$ and B are edge-transitive and one is arc-transitive, then $A \times B$ is edge-transitive.*

We conjecture that the converse also holds.

2.e. Edge-transitivity in strong products of finite graphs

Theorem *The strong product $G = A \boxtimes B$ of two connected, non-trivial graphs is edge-transitive if and only if both factors are complete.*

2.f. The weak Cartesian product

Let G_ι, $\iota \in I$, be a set of graphs.

Then the Cartesian product $G = \Box_{\iota \in I} G_\iota$ is defined as follows:

$V(G)$ is the set of all functions $x : \iota \mapsto x_\iota$, with $x_\iota \in V(G_\iota)$.

Two vertices x and y are adjacent in G if there exist a $\kappa \in I$ such that $x_\kappa y_\kappa \in E(G_\kappa)$ and $x_\iota = y_\iota$ for $\iota \in I \setminus \kappa$.

If I is finite, then G is connected if and only if all G_ι are connected.

If I is infinite and if the G_ι are non-trivial, then G is disconnected, even if all factors G_ι are connected.

A weak Cartesian product of graphs G_ι, $\iota \in I$, is a connected component of the Cartesian product of the G_ι.

To identify the component it suffices to specify a vertex in it, say a.

We write $\square^a_{\iota\in I} G_\iota$ for the connected component of G that contains a.

Clearly $V(\square^a_{\iota\in I} G_\iota)$ consists of a and all vertices of $\square_{\iota\in I} G_\iota$ that differ from a in only finitely many coordinates.

Every connected graph has a unique representation as a weak Cartesian product of prime graphs*.

*This and the following proposition are (independently) from D. Miller and I. They date back to the early 1970's. For details compare the Handbook of Product Graphs by Hammack, Imrich and Klavžar, CRC Press 2011.

Proposition *Let* $G = \square^{a}_{\iota \in I} G_\iota$*, where the* G_ι*,* $\iota \in I$*, are connected and prime. Then every* $\varphi \in \mathrm{Aut}(G)$ *is of the form*

$$\varphi(x)_\iota = \varphi_\iota(x_{\pi(\iota)}),$$

where π *is a permutation of* I *and* $\varphi_\iota \in \mathrm{Aut}(G_\iota)$ *for all* $\iota \in I$*. Furthermore,* $\varphi(a)_\iota \neq a_\iota$ *for only finitely many* ι*.*

The condition that $\varphi(a)_\iota \neq a_\iota$ for only finitely many ι always holds if I is finite. Then $\varphi(x)_\iota = \varphi_\iota(x_{\pi(\iota)})$ completely describes $\mathrm{Aut}(G)$.

The replacements $\pi^{-1} \to \psi$, $x \to \alpha$, $\iota \to \lambda$ and $\varphi \to t$ transform this formula into $\alpha^t\lambda = (\alpha\lambda)^{\psi t}$, a formula from this morning's lecture by Colva Roney-Dougal. Her example for it was $K_3 \,\square\, K_2$.

She also showed that Aut($K_3 \,\square\, K_2$) $\cong$ Aut($K_3 \cup K_2$). For finite I (and prime, connected G_ι) this also holds in general:

$$\mathrm{Aut}(\mathop{\square}_{\iota\in I} G_\iota) \cong \mathrm{Aut}(\bigcup_{\iota\in I} G_\iota).$$

Theorem *The weak Cartesian product $G = \square^a_{\iota\in I} G_\iota$ of connected, prime graphs G_ι is vertex-transitive under the following necessary and sufficient condition:*

If v is a vertex of a G_κ that is not vertex-transitive, then there is an infinite set $K \subseteq I$, and isomorphisms $\varphi_{\lambda,\kappa} : G_\lambda \to G_\kappa$ for every $\lambda \in K$, with $\varphi_{\lambda,\kappa}(a_\lambda) = v$.

Corollary *If all* G_ι, $\iota \in I$, *are vertex-transitive, then* $G = \square_{\iota\in I}^{a} G_\iota$ *is also vertex transitive.*

Corollary *If* I *is finite, then* G *is v.t. if and only if all* G_ι, $\iota \in I$, *are v.t.*

Lemma *Let* H *be a connected, prime graph with two vertex-orbits, and* $G = \square_{\iota\in I}^{a} G_\iota$, *where* $G_\iota \cong H$ *and* $\aleph_0 \leq |I|$. *Then* G *is v.t. or has infinitely many vertex-orbits.*

Lemma *Let H be a connected, e.t. and v.t.*
Consider $G = \square_{\iota\in I}^{a} H_\iota$, where $H_\iota \cong H$ and $2 \leq |I|$.
Then G is also e.t.and v.t.
Furthermore, G is half-transitive if and only if H is half-transitive.

Lemma *Let H be connected, e.t. but not v.t., with vertex orbits V_1 and V_2.*
Consider $G = \square_{\iota\in I}^{a} H_\iota$, $H_\iota \cong H$, $2 \leq |I|$, where infinitely many a_ι are in V_1, and infinitely many in V_2.
Then G is e.t. (but only half-transitive) and v.t.

Theorem *Let G be a connected, e.t. graph that is not prime with respect to the Cartesian product.*

Then G is the Cartesian or weak Cartesian power of a connected, e.t. graph H.

G is always v.t, even when H is not.

3. Primitive Permutation Groups*

Let G be a permutation group acting on a set V.

A partition π of V is distinguishing if the only element of G that fixes each cell of π is the identity.

The distinguishing number of G is the minimum number of cells in a distinguishing partition.

Serres† proved 1997 that if G is primitive on V and $|V| \geq 32$, then its distinguishing number is 2. He used the classification of finite simple groups.

*C. Godsil, Distinguishing primitive permutation groups, arXiv:0806.2078v2, 2009

†Ákos Seress. Primitive groups with no regular orbits on the set of subsets. Bull. London Math. Soc. 29 (1997), 697-704.

Godsil does it differently, the bound is then 336. He also needs the classification.

But, as Godsil's proof uses results that invoke Cartesian powers of Johnson graphs a short outline is presented.

The proof uses four main results. First a few definitions.

The degree of a permutation group G on a set V is the size of V, and the minimum degree of G is the minimum number of points moved by a nonidentity element of G.

The minimum degree of G is also known as the motion $\mathrm{m}(G)$ of G.

Godsil proves that if G is a permutation group with distinguishing number at least 3 and the minimum degree of G is δ, then

$$|G| \geq 1 + 2^{\delta/2}.$$

This is also known as the Motion Lemma[*] and usually stated in the form that G has distinguishing number ≤ 2 if

$$2^{\mathrm{m}(G)/2} \geq |G|.$$

[*] A. Russell and R. Sundaram, A note on the asymptotics and computational complexity of graph distinguishability, Electron. J. Combin. 5 (1998), R23.

The Johnson graph $J(m, \ell)$ has the ℓ subsets of $\{1, ..., m\}$ as its vertices, and two ℓ-subsets are adjacent if their intersection has size $\ell - 1$. $J(m, \ell)$ and $J(m, m - \ell)$ are isomorphic, and so we will assume that $2\ell \leq m$. The complete graphs are Johnson graphs (with $\ell = 1$).

The next two results invoked by Godsil also depend on the classification of finite simple groups.

Theorem (Guralnick and Magaard 1998*) *Let G be a primitive permutation group of degree v. If the minimal degree of G is at most $v/2$, then one the following holds:*

(a) G is affine over $GF(2)^\dagger$ and its minimal degree is $v/2$.

(b) G is a transitive subgroup of the automorphism group of the Cartesian power of a Johnson graph.

*On the minimal degree of a primitive permutation group, J. Algebra 207 (1998), 127-145.

†For a definition see today's lecture of Colva Roney-Dougal.

Theorem (Maróti 2002*) *Let G be a primitive permutation group of degree v. Then one of the following holds:*

(a) G is a transitive subgroup of the automorphism group of the Cartesian power of a Johnson graph.

(b) $v \in \{11, 12, 23, 24\}$ and G is one of the Mathieu groups.

(c) $|G| < v^{1+\lceil \log_2 v \rceil}$.

*On the orders of primitive groups, J. Algebra 258 (2002), 631-640.

Klavžar and I proved 2006* that the distinguishing number of the product of two complete graphs is 2, except for K_2^2, K_2^3 and K_3^2.

Putting it all together one obtains relatively quickly that if G is primitive and on V and if $|V| \geq 336$, then the distinguishing number of G is 2.

*Distinguishing Cartesian powers of graphs, J. Graph Theory 53 (2006), 250-260.

4. The Infinite Motion Conjecture

We conclude with a conjecture that generalizes the Motion Lemma.

One says a graph G is 2-distinguishable if $\mathsf{Aut}(G)$ has distinguishing number ≤ 2, and defines the motion $\mathsf{m}(G)$ of G as $\mathsf{m}(\mathsf{Aut}(G))$.

By the Motion Lemma a finite graph G is then 2-distinguishable if $2^{\mathsf{m}(G)/2} \geq |\,\mathsf{Aut}(G))|$.

Suppose $G \in \Gamma$ has infinite motion, i.e. $\mathsf{m}(G) = \aleph_0$. Because the size of the automorphism group of a graph in Γ is at most $2^{\aleph_0}$, we have

$$2^{\mathsf{m}(G)/2} = 2^{\aleph_0} \geq |\,\mathsf{Aut}(G)|.$$

If the Motion Lemma also held for infinite graphs, we could conclude that G is 2-distinguishable.

This is the Infinite Motion Conjecture of Tom Tucker:

Conjecture (Tucker 2011*) *Every graph in* Γ *with infinite motion is 2-distinguishable.*

This conjecture is still open despite many interesting and deep partial results.

*Distinguishing Maps, Electron. J. Combin. 18 (2011), R50.

THE END