An infinite-dimensional $\square_q$-module obtained from the q-shuffle algebra for affine $\mathfrak{sl}_2$

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Overview

We will first recall the notion of a **tridiagonal pair**.

We will give three examples of a tridiagonal pair, using representations of the **Onsager algebra**, the **positive part of** $U_q(\widehat{\mathfrak{sl}}_2)$, and the q-**Onsager algebra**.

Motivated by these algebras we will bring in an algebra $\square_q$.

We will introduce an infinite-dimensional $\square_q$-module, said to be **NIL**.

We will describe the NIL $\square_q$-module from sixteen points of view.

In this description we will use the **free algebra** $\mathbb{V}$ on two generators, as well as a q-**shuffle algebra** structure on $\mathbb{V}$.

The concept of a **tridiagonal pair** was introduced in 1999 by Tatsuro Ito, Kenichiro Tanabe, and Paul Terwilliger.

This concept is defined as follows.

Let $\mathbb{F}$ denote a field.

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension.

Consider two $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$.

The definition of a tridiagonal pair

The above pair A, A^* is called a **tridiagonal pair** whenever:

(i) each of A, A^* is diagonalizable;

(ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

(iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

(iv) there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

Definition of a tridiagonal pair, cont.

Referring to the above definition, it turns out that $d = \delta$.

This common value is called the **diameter** of the pair.

The eigenvalues of a tridiagonal pair

Refer to the above tridiagonal pair A, A^*.

For $0 \leq i \leq d$, let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) for the eigenspace V_i (resp. V^*_i).

The sequence $\{\theta_i\}_{i=0}^{d}$ (resp. $\{\theta^*_i\}_{i=0}^{d}$) is an ordering of the eigenvalues of A (resp. A^*).

This ordering is called **standard**.

Three examples of a tridiagonal pair

We now give some examples of a tridiagonal pair.

Our examples come from representation theory.

We will consider some representations of the following three algebras:

- The Onsager algebra $\mathcal{O}$;
- The positive part U_q^+ of $U_q(\widehat{\mathfrak{sl}}_2)$;
- The q-Onsager algebra $\mathcal{O}_q$.

The Onsager algebra $\mathcal{O}$

The **Onsager algebra** $\mathcal{O}$ is the Lie algebra over $\mathbb{C}$ defined by generators A, A^* and relations

$$[A, [A, [A, A^*]]] = 4[A, A^*],$$
$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A].$$

The above equations are called the **Dolan/Grady relations**.

The Onsager algebra $\mathcal{O}$, cont.

Let V denote a finite-dimensional irreducible $\mathcal{O}$-module.

Then the $\mathcal{O}$-generators A, A^* act on V as a tridiagonal pair.

For this tridiagonal pair the eigenvalues of A and A^* look as follows in standard order:

$$d - 2i \qquad\qquad (0 \leq i \leq d).$$

From now on, fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2, \ldots$$

Let U_q^+ denote the associative $\mathbb{F}$-algebra defined by generators A, A^* and relations

$$A^3 A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 = 0,$$

$$A^{*3} A - [3]_q A^{*2} A A^* + [3]_q A^* A A^{*2} - A A^{*3} = 0.$$

The above equations are called the q-**Serre relations**.

We call U_q^+ the **positive part of** $U_q(\widehat{\mathfrak{sl}}_2)$.

The positive part U_q^+, cont.

Let V denote a finite-dimensional irreducible U_q^+-module on which the U_q^+-generators A, A^* are not nilpotent.

Then A, A^* act on V as a tridiagonal pair.

For this tridiagonal pair the eigenvalues of A and A^* look as follows in standard order:

$$A: \qquad aq^{d-2i} \qquad (0 \leq i \leq d),$$
$$A^*: \qquad bq^{d-2i} \qquad (0 \leq i \leq d).$$

The scalars a, b depend on the U_q^+-module V.

The q-Onsager algebra $\mathcal{O}_q$

Let $\mathcal{O}_q$ denote the associative $\mathbb{F}$-algebra defined by generators A, A^* and relations

$$\begin{aligned} A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 \\ = (q^2 - q^{-2})^2(A^*A - AA^*), \end{aligned}$$

$$\begin{aligned} A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} \\ = (q^2 - q^{-2})^2(AA^* - A^*A). \end{aligned}$$

The above equations are called the q-**Dolan/Grady relations**.

We call $\mathcal{O}_q$ the q-**Onsager algebra**.

A bit of history

The q-Dolan/Grady relations first appeared in Algebraic Combinatorics, in the study of Q-polynomial distance-regular graphs (Terwilliger 1993).

The q-Onsager algebra was formally introduced by Terwilliger in 2003.

Starting around 2005, Pascal Baseilhac applied the q-Onsager algebra to Integrable Systems.

Let V denote a finite-dimensional irreducible $\mathcal{O}_q$-module on which the $\mathcal{O}_q$-generators A, A^* are diagonalizable.

Then A, A^* act on V as a tridiagonal pair. For this pair the eigenvalues of A and A^* look as follows in standard order:

$$
\begin{aligned}
A: &\quad aq^{d-2i} + a^{-1}q^{2i-d} \qquad (0 \le i \le d), \\
A^*: &\quad bq^{d-2i} + b^{-1}q^{2i-d} \qquad (0 \le i \le d).
\end{aligned}
$$

The scalars a, b depend on the $\mathcal{O}_q$-module V.

Comparing U_q^+ and $\mathcal{O}_q$

Consider how the algebras U_q^+ and $\mathcal{O}_q$ are related.

These algebras have at least a superficial resemblance, since for the q-Serre relations and q-Dolan/Grady relations their left-hand sides match.

We now consider how U_q^+ and $\mathcal{O}_q$ are related at an algebraic level.

To do this, we bring in another algebra $\square_q$.

Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

The algebra $\Box_q$

Definition

Let $\Box_q$ denote the associative $\mathbb{F}$-algebra with generators $\{x_i\}_{i \in \mathbb{Z}_4}$ and relations

$$\frac{q x_i x_{i+1} - q^{-1} x_{i+1} x_i}{q - q^{-1}} = 1,$$

$$x_i^3 x_{i+2} - [3]_q x_i^2 x_{i+2} x_i + [3]_q x_i x_{i+2} x_i^2 - x_{i+2} x_i^3 = 0.$$

The algebra $\square_q$ has the following $\mathbb{Z}_4$ symmetry.

There exists an automorphism ρ of $\square_q$ that sends $x_i \mapsto x_{i+1}$ for $i \in \mathbb{Z}_4$. Moreover $\rho^4 = 1$.

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The algebra $\square_q$ is related to U_q^+ in the following way.

Definition

Define the subalgebras $\square_q^{\text{even}}$, $\square_q^{\text{odd}}$ of $\square_q$ such that

(i) $\square_q^{\text{even}}$ is generated by x_0, x_2;

(ii) $\square_q^{\text{odd}}$ is generated by x_1, x_3.

Theorem

The following (i)–(iii) *hold:*

(i) *there exists an* $\mathbb{F}$*-algebra isomorphism* $U_q^+ \to \square_q^{\text{even}}$ *that sends* $A \mapsto x_0$ *and* $A^* \mapsto x_2$;

(ii) *there exists an* $\mathbb{F}$*-algebra isomorphism* $U_q^+ \to \square_q^{\text{odd}}$ *that sends* $A \mapsto x_1$ *and* $A^* \mapsto x_3$;

(iii) *the following is an isomorphism of* $\mathbb{F}$*-vector spaces:*

$$\begin{array}{ccc} \square_q^{\text{even}} \otimes \square_q^{\text{odd}} & \to & \square_q \\ u \otimes v & \mapsto & uv \end{array}$$

We just showed how the vector space $\square_q$ is isomorphic to $U_q^+ \otimes U_q^+$.

We now describe how $\square_q$ is related to the q-Onsager algebra $\mathcal{O}_q$.

Theorem

Pick nonzero $a, b \in \mathbb{F}$. Then there exists a unique $\mathbb{F}$-algebra homomorphism $\natural : \mathcal{O}_q \to \square_q$ that sends

$$A \mapsto a x_0 + a^{-1} x_1, \qquad B \mapsto b x_2 + b^{-1} x_3.$$

The homomorphism $\natural$ is injective.

Motivated by the previous theorem, we wish to better understand the algebra $\square_q$.

So we consider the $\square_q$-modules.

The finite-dimensional irreducible $\square_q$-modules were classified up to isomorphism by Yang Yang 2017.

Our topic here is a certain infinite-dimensional $\square_q$-module, said to be NIL.

The NIL $\square_q$-modules

Definition

Let V denote a $\square_q$-module. A vector $\xi \in V$ is called NIL whenever $x_1\xi = 0$ and $x_3\xi = 0$ and $\xi \neq 0$.

Definition

A $\square_q$-module V is called NIL whenever V is generated by a NIL vector.

The NIL $\square_q$-module $\mathbf{U}$

Theorem

Up to isomorphism, there exists a unique NIL $\square_q$-module, which we denote by $\mathbf{U}$.

The $\square_q$-module $\mathbf{U}$ is irreducible and infinite-dimensional.

The NIL $\square_q$-module $\mathbf{U}$

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Theorem

The $\square_q$-module $\mathbf{U}$ has a unique sequence of subspaces $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ such that

(i) $\mathbf{U}_0 \neq 0$;

(ii) *the sum* $\mathbf{U} = \sum_{n\in\mathbb{N}} \mathbf{U}_n$ *is direct;*

(iii) *for* $n \in \mathbb{N}$,

$$x_0\mathbf{U}_n \subseteq \mathbf{U}_{n+1}, \qquad x_1\mathbf{U}_n \subseteq \mathbf{U}_{n-1},$$
$$x_2\mathbf{U}_n \subseteq \mathbf{U}_{n+1}, \qquad x_3\mathbf{U}_n \subseteq \mathbf{U}_{n-1},$$

where $\mathbf{U}_{-1} = 0$.

The NIL $\square_q$-module $\mathbf{U}$, cont.

Theorem

The sequence $\{\mathbf{U}_n\}_{n\in\mathbb{N}}$ is described as follows.

The subspace $\mathbf{U}_0$ has dimension 1.

The nonzero vectors in $\mathbf{U}_0$ are precisely the NIL vectors in $\mathbf{U}$, and each of these vectors generates $\mathbf{U}$.

Let ξ denote a NIL vector in $\mathbf{U}$. Then for $n \in \mathbb{N}$, the subspace $\mathbf{U}_n$ is spanned by the vectors

$$u_1 u_2 \cdots u_n \xi, \qquad u_i \in \{x_0, x_2\}, \qquad 1 \leq i \leq n.$$

Shortly we will describe the $\square_q$-module $\mathbf{U}$ in more detail.

To prepare, we comment on free algebras and q-shuffle algebras.

The free algebra $\mathbb{V}$

From now on, $\mathbb{V}$ denotes the free associative $\mathbb{F}$-algebra on two generators A, B.

For $n \in \mathbb{N}$, a **word of length** n in $\mathbb{V}$ is a product $v_1 v_2 \cdots v_n$ such that $v_i \in \{A, B\}$ for $1 \leq i \leq n$.

The **standard basis** for $\mathbb{V}$ consists of the words.

A bilinear form on $\mathbb{V}$

There exists a symmetric bilinear form $(\, , \,) : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$ with respect to which the standard basis is orthonormal.

Recall that the algebra $\mathrm{End}(\mathbb{V})$ consists of the $\mathbb{F}$-linear maps from $\mathbb{V}$ to $\mathbb{V}$.

For $X \in \mathrm{End}(\mathbb{V})$ there exists a unique $X^* \in \mathrm{End}(\mathbb{V})$ such that $(Xu, v) = (u, X^* v)$ for all $u, v \in \mathbb{V}$.

The element X^* is called the **adjoint of** X with respect to $(\, , \,)$.

We define an invertible $K \in \mathrm{End}(\mathbb{V})$ as follows.

Definition

The map K is the automorphism of the free algebra $\mathbb{V}$ that sends $A \mapsto q^2 A$ and $B \mapsto q^{-2} B$.

We have $K^* = K$.

The automorphism K of $\mathbb{V}$, cont.

The map K acts on the standard basis for $\mathbb{V}$ in the following way.

For a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$K(v) = vq^{\langle v_1, A\rangle + \langle v_2, A\rangle + \cdots + \langle v_n, A\rangle},$$
$$K^{-1}(v) = vq^{\langle v_1, B\rangle + \langle v_2, B\rangle + \cdots + \langle v_n, B\rangle}$$

where

$\langle\, , \rangle$	A	B
A	2	-2
B	-2	2

Left and right multiplication in $\mathbb{V}$

Definition

We define four maps in $\mathrm{End}(\mathbb{V})$, denoted

$$A_L, \qquad B_L, \qquad A_R, \qquad B_R.$$

For $v \in \mathbb{V}$,

$$A_L(v) = Av, \qquad B_L(v) = Bv, \qquad A_R(v) = vA, \qquad B_R(v) = vB.$$

Some adjoints

We now consider

$$A_L^*, \qquad B_L^*, \qquad A_R^*, \qquad B_R^*.$$

Lemma

For a word $v = v_1 v_2 \cdots v_n$ *in* $\mathbb{V}$,

$$A_L^*(v) = v_2 \cdots v_n \delta_{v_1,A}, \qquad B_L^*(v) = v_2 \cdots v_n \delta_{v_1,B},$$
$$A_R^*(v) = v_1 \cdots v_{n-1} \delta_{v_n,A}, \qquad B_R^*(v) = v_1 \cdots v_{n-1} \delta_{v_n,B}.$$

The q-shuffle algebra $\mathbb{V}$

We have been discussing the free algebra $\mathbb{V}$.

There is another algebra structure on $\mathbb{V}$, called the q-**shuffle algebra**. This is due to M. Rosso 1995.

The q-shuffle product will be denoted by $\star$.

For $X \in \{A, B\}$ and a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$X \star v = \sum_{i=0}^{n} v_1 \cdots v_i X v_{i+1} \cdots v_n q^{\langle v_1, X\rangle + \langle v_2, X\rangle + \cdots + \langle v_i, X\rangle},$$

$$v \star X = \sum_{i=0}^{n} v_1 \cdots v_i X v_{i+1} \cdots v_n q^{\langle v_n, X\rangle + \langle v_{n-1}, X\rangle + \cdots + \langle v_{i+1}, X\rangle}.$$

The map K is an automorphism of the q-shuffle algebra $\mathbb{V}$.

The q-shuffle algebra $\mathbb{V}$, cont.

Definition

We define four maps in $\mathrm{End}(\mathbb{V})$, denoted

$$A_\ell, \qquad B_\ell, \qquad A_r, \qquad B_r.$$

For $v \in \mathbb{V}$,

$$A_\ell(v) = A \star v, \quad B_\ell(v) = B \star v, \quad A_r(v) = v \star A, \quad B_r(v) = v \star B.$$

Some more adjoints

We now consider

$$A_\ell^*, \qquad B_\ell^*, \qquad A_r^*, \qquad B_r^*.$$

Lemma

For a word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$A_\ell^*(v) = \sum_{i=0}^{n} v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \delta_{v_i,A} q^{\langle v_1,A\rangle + \langle v_2,A\rangle + \cdots + \langle v_{i-1},A\rangle},$$

$$B_\ell^*(v) = \sum_{i=0}^{n} v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \delta_{v_i,B} q^{\langle v_1,B\rangle + \langle v_2,B\rangle + \cdots + \langle v_{i-1},B\rangle},$$

$$A_r^*(v) = \sum_{i=0}^{n} v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \delta_{v_i,A} q^{\langle v_n,A\rangle + \langle v_{n-1},A\rangle + \cdots + \langle v_{i+1},A\rangle},$$

$$B_r^*(v) = \sum_{i=0}^{n} v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \delta_{v_i,B} q^{\langle v_n,B\rangle + \langle v_{n-1},B\rangle + \cdots + \langle v_{i+1},B\rangle}.$$

Comparing the free algebra and the q-shuffle algebra

We now compare the free algebra $\mathbb{V}$ with the q-shuffle algebra $\mathbb{V}$.

To do this, we recall the concept of a derivation.

Let $\mathcal{A}$ denote an associative $\mathbb{F}$-algebra, and let φ, ϕ denote automorphisms of $\mathcal{A}$.

By a (φ, ϕ)-**derivation** of $\mathcal{A}$ we mean an $\mathbb{F}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ such that for all $u, v \in \mathcal{A}$,

$$\delta(uv) = \varphi(u)\delta(v) + \delta(u)\phi(v).$$

The following two lemmas are due to M. Rosso and J. Green 1995.

Lemma

For the free algebra $\mathbb{V}$,

(i) A_ℓ^* *is a* (K, I)*-derivation;*

(ii) B_ℓ^* *is a* (K^{-1}, I)*-derivation;*

(iii) A_r^* *is a* (I, K)*-derivation;*

(iv) B_r^* *is a* (I, K^{-1})*-derivation.*

Comparing the free algebra and the q-shuffle algebra

Lemma

For the q-shuffle algebra $\mathbb{V}$,

(i) A_L^* *is a* (K, I)*-derivation;*

(ii) B_L^* *is a* (K^{-1}, I)*-derivation;*

(iii) A_R^* *is a* (I, K)*-derivation;*

(iv) B_R^* *is a* (I, K^{-1})*-derivation.*

Some relations

We will need some relations satisfied by K, K^{-1} and

$$A_L^*, \quad B_L^*, \quad A_R^*, \quad B_R^*, \quad A_\ell, \quad B_\ell, \quad A_r, \quad B_r.$$

We acknowledge that these relations are already known to the experts, such as Kashiwara 1991, Rosso 1995, Green 1995.

Some relations

Theorem

We have

$$KA_L^* = q^{-2}A_L^*K, \qquad KB_L^* = q^2 B_L^* K,$$
$$KA_R^* = q^{-2}A_R^*K, \qquad KB_R^* = q^2 B_R^* K,$$

$$KA_\ell = q^2 A_\ell K, \qquad KB_\ell = q^{-2} B_\ell K,$$
$$KA_r = q^2 A_r K, \qquad KB_r = q^{-2} B_r K,$$

$$A_L^* A_R^* = A_R^* A_L^*, \qquad B_L^* B_R^* = B_R^* B_L^*,$$
$$A_L^* B_R^* = B_R^* A_L^*, \qquad B_L^* A_R^* = A_R^* B_L^*,$$

$$A_\ell A_r = A_r A_\ell, \qquad B_\ell B_r = B_r B_\ell,$$
$$A_\ell B_r = B_r A_\ell, \qquad B_\ell A_r = A_r B_\ell,$$

Some relations, cont.

Theorem

We have

$$A_L^* B_r = B_r A_L^*, \qquad B_L^* A_r = A_r B_L^*,$$
$$A_R^* B_\ell = B_\ell A_R^*, \qquad B_R^* A_\ell = A_\ell B_R^*,$$

$$A_L^* B_\ell = q^{-2} B_\ell A_L^*, \qquad B_L^* A_\ell = q^{-2} A_\ell B_L^*,$$
$$A_R^* B_r = q^{-2} B_r A_R^*, \qquad B_R^* A_r = q^{-2} A_r B_R^*,$$

$$A_L^* A_\ell - q^2 A_\ell A_L^* = I, \qquad A_R^* A_r - q^2 A_r A_R^* = I,$$
$$B_L^* B_\ell - q^2 B_\ell B_L^* = I, \qquad B_R^* B_r - q^2 B_r B_R^* = I,$$

$$A_L^* A_r - A_r A_L^* = K, \qquad B_L^* B_r - B_r B_L^* = K^{-1},$$
$$A_R^* A_\ell - A_\ell A_R^* = K, \qquad B_R^* B_\ell - B_\ell B_R^* = K^{-1},$$

Some relations, cont.

Theorem

We have

$$
\begin{aligned}
A_\ell^3 B_\ell - [3]_q A_\ell^2 B_\ell A_\ell + [3]_q A_\ell B_\ell A_\ell^2 - B_\ell A_\ell^3 &= 0,\\
B_\ell^3 A_\ell - [3]_q B_\ell^2 A_\ell B_\ell + [3]_q B_\ell A_\ell B_\ell^2 - A_\ell B_\ell^3 &= 0,\\
A_r^3 B_r - [3]_q A_r^2 B_r A_r + [3]_q A_r B_r A_r^2 - B_r A_r^3 &= 0,\\
B_r^3 A_r - [3]_q B_r^2 A_r B_r + [3]_q B_r A_r B_r^2 - A_r B_r^3 &= 0.
\end{aligned}
$$

Some more relations

Applying the adjoint map to the above relations, we obtain the following relations satisfied by K, K^{-1} and

$$A_L, \quad B_L, \quad A_R, \quad B_R, \quad A^*_\ell, \quad B^*_\ell, \quad A^*_r, \quad B^*_r.$$

Some more relations

Theorem

We have

$$KA_L = q^2 A_L K, \qquad KB_L = q^{-2} B_L K,$$
$$KA_R = q^2 A_R K, \qquad KB_R = q^{-2} B_R K,$$

$$KA_\ell^* = q^{-2} A_\ell^* K, \qquad KB_\ell^* = q^2 B_\ell^* K,$$
$$KA_r^* = q^{-2} A_r^* K, \qquad KB_r^* = q^2 B_r^* K,$$

$$A_L A_R = A_R A_L, \qquad B_L B_R = B_R B_L,$$
$$A_L B_R = B_R A_L, \qquad B_L A_R = A_R B_L,$$

$$A_\ell^* A_r^* = A_r^* A_\ell^*, \qquad B_\ell^* B_r^* = B_r^* B_\ell^*,$$
$$A_\ell^* B_r^* = B_r^* A_\ell^*, \qquad B_\ell^* A_r^* = A_r^* B_\ell^*,$$

Theorem

We have

$$A_L B_r^* = B_r^* A_L, \qquad B_L A_r^* = A_r^* B_L,$$
$$A_R B_\ell^* = B_\ell^* A_R, \qquad B_R A_\ell^* = A_\ell^* B_R,$$

$$A_L B_\ell^* = q^2 B_\ell^* A_L, \qquad B_L A_\ell^* = q^2 A_\ell^* B_L,$$
$$A_R B_r^* = q^2 B_r^* A_R, \qquad B_R A_r^* = q^2 A_r^* B_R,$$

$$A_\ell^* A_L - q^2 A_L A_\ell^* = I, \qquad A_r^* A_R - q^2 A_R A_r^* = I,$$
$$B_\ell^* B_L - q^2 B_L B_\ell^* = I, \qquad B_r^* B_R - q^2 B_R B_r^* = I,$$

$$A_r^* A_L - A_L A_r^* = K, \qquad B_r^* B_L - B_L B_r^* = K^{-1},$$
$$A_\ell^* A_R - A_R A_\ell^* = K, \qquad B_\ell^* B_R - B_R B_\ell^* = K^{-1},$$

Some more relations, cont.

Theorem

We have

$$(A_\ell^*)^3 B_\ell^* - [3]_q (A_\ell^*)^2 B_\ell^* A_\ell^* + [3]_q A_\ell^* B_\ell^* (A_\ell^*)^2 - B_\ell^* (A_\ell^*)^3 = 0,$$
$$(B_\ell^*)^3 A_\ell^* - [3]_q (B_\ell^*)^2 A_\ell^* B_\ell^* + [3]_q B_\ell^* A_\ell^* (B_\ell^*)^2 - A_\ell^* (B_\ell^*)^3 = 0,$$
$$(A_r^*)^3 B_r^* - [3]_q (A_r^*)^2 B_r^* A_r^* + [3]_q A_r^* B_r^* (A_r^*)^2 - B_r^* (A_r^*)^3 = 0,$$
$$(B_r^*)^3 A_r^* - [3]_q (B_r^*)^2 A_r^* B_r^* + [3]_q B_r^* A_r^* (B_r^*)^2 - A_r^* (B_r^*)^3 = 0.$$

The 2-sided ideal J of the free algebra $\mathbb{V}$

Let J denote the 2-sided ideal of the free algebra $\mathbb{V}$ generated by

$$J^+ = A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3,$$
$$J^- = B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3.$$

The quotient algebra $\mathbb{V}/J$ is isomorphic to U_q^+.

The 2-sided ideal J of the free algebra $\mathbb{V}$, cont.

Lemma

The subspace J is invariant under $K^{\pm 1}$ and

$$A_L, \quad B_L, \quad A_R, \quad B_R, \quad A^*_\ell, \quad B^*_\ell, \quad A^*_r, \quad B^*_r.$$

On the quotient $\mathbb{V}/J$,

$$A_L^3 B_L - [3]_q A_L^2 B_L A_L + [3]_q A_L B_L A_L^2 - B_L A_L^3 = 0,$$
$$B_L^3 A_L - [3]_q B_L^2 A_L B_L + [3]_q B_L A_L B_L^2 - A_L B_L^3 = 0,$$
$$A_R^3 B_R - [3]_q A_R^2 B_R A_R + [3]_q A_R B_R A_R^2 - B_R A_R^3 = 0,$$
$$B_R^3 A_R - [3]_q B_R^2 A_R B_R + [3]_q B_R A_R B_R^2 - A_R B_R^3 = 0.$$

Let U denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ generated by A, B.

The algebra U is isomorphic to U_q^+ (Rosso 1995).

The subalgebra U of the q-shuffle algebra V, cont.

Lemma

The subspace U is invariant under $K^{\pm 1}$ and

$$A^*_L, \quad B^*_L, \quad A^*_R, \quad B^*_R, \quad A_\ell, \quad B_\ell, \quad A_r, \quad B_r.$$

On U,

$$(A^*_L)^3 B^*_L - [3]_q (A^*_L)^2 B^*_L A^*_L + [3]_q A^*_L B^*_L (A^*_L)^2 - B^*_L (A^*_L)^3 = 0,$$
$$(B^*_L)^3 A^*_L - [3]_q (B^*_L)^2 A^*_L B^*_L + [3]_q B^*_L A^*_L (B^*_L)^2 - A^*_L (B^*_L)^3 = 0,$$
$$(A^*_R)^3 B^*_R - [3]_q (A^*_R)^2 B^*_R A^*_R + [3]_q A^*_R B^*_R (A^*_R)^2 - B^*_R (A^*_R)^3 = 0,$$
$$(B^*_R)^3 A^*_R - [3]_q (B^*_R)^2 A^*_R B^*_R + [3]_q B^*_R A^*_R (B^*_R)^2 - A^*_R (B^*_R)^3 = 0.$$

The main results

We are now ready to state our main results, which are about the $\square_q$-module $\mathbf{U}$.

For notational convenience define $Q = 1 - q^2$.

The main results

Theorem

For each row in the tables below, the vector space $\mathbb{V}/J$ becomes a $\square_q$-module on which the generators $\{x_i\}_{i\in\mathbb{Z}_4}$ act as indicated.

module label	x_0	x_1	x_2	x_3
I	A_L	$Q(A^*_\ell - B^*_r K)$	B_L	$Q(B^*_\ell - A^*_r K^{-1})$
IS	A_R	$Q(A^*_r - B^*_\ell K)$	B_R	$Q(B^*_r - A^*_\ell K^{-1})$
IT	B_L	$Q(B^*_\ell - A^*_r K^{-1})$	A_L	$Q(A^*_\ell - B^*_r K)$
IST	B_R	$Q(B^*_r - A^*_\ell K^{-1})$	A_R	$Q(A^*_r - B^*_\ell K)$

module label	x_0	x_1	x_2	x_3
II	$Q(A_L - KB_R)$	A^*_ℓ	$Q(B_L - K^{-1}A_R)$	B^*_ℓ
IIS	$Q(A_R - KB_L)$	A^*_r	$Q(B_R - K^{-1}A_L)$	B^*_r
IIT	$Q(B_L - K^{-1}A_R)$	B^*_ℓ	$Q(A_L - KB_R)$	A^*_ℓ
IIST	$Q(B_R - K^{-1}A_L)$	B^*_r	$Q(A_R - KB_L)$	A^*_r

Each $\square_q$-module in the tables is isomorphic to $\mathbf{U}$.

The main results, cont.

Theorem

For each row in the tables below, the vector space U becomes a $\square_q$-module on which the generators $\{x_i\}_{i\in\mathbb{Z}_4}$ act as indicated.

module label	x_0	x_1	x_2	x_3
III	A_ℓ	$Q(A_L^* - B_R^* K)$	B_ℓ	$Q(B_L^* - A_R^* K^{-1})$
IIIS	A_r	$Q(A_R^* - B_L^* K)$	B_r	$Q(B_R^* - A_L^* K^{-1})$
IIIT	B_ℓ	$Q(B_L^* - A_R^* K^{-1})$	A_ℓ	$Q(A_L^* - B_R^* K)$
IIIST	B_r	$Q(B_R^* - A_L^* K^{-1})$	A_r	$Q(A_R^* - B_L^* K)$

module label	x_0	x_1	x_2	x_3
IV	$Q(A_\ell - KB_r)$	A_L^*	$Q(B_\ell - K^{-1}A_r)$	B_L^*
IVS	$Q(A_r - KB_\ell)$	A_R^*	$Q(B_r - K^{-1}A_\ell)$	B_R^*
IVT	$Q(B_\ell - K^{-1}A_r)$	B_L^*	$Q(A_\ell - KB_r)$	A_L^*
IVST	$Q(B_r - K^{-1}A_\ell)$	B_R^*	$Q(A_r - KB_\ell)$	A_R^*

Each $\square_q$-module in the tables is isomorphic to $\mathbf{U}$.

The main results, cont.

Theorem

For the above $\square_q$-modules on $\mathbb{V}/J$, the elements x_1 and x_3 act on the algebra $\mathbb{V}/J$ as a derivation of the following sort:

module label	x_1	x_3
I, II	(K, I)-derivation	(K^{-1}, I)-derivation
IS, IIS	(I, K)-derivation	(I, K^{-1})-derivation
IT, IIT	(K^{-1}, I)-derivation	(K, I)-derivation
IST, IIST	(I, K^{-1})-derivation	(I, K)-derivation

Theorem

For the above $\square_q$-modules on U, the elements x_1 and x_3 act on the algebra U as a derivation of the following sort:

module label	x_1	x_3
III, IV	(K, I)-derivation	(K^{-1}, I)-derivation
IIIS, IVS	(I, K)-derivation	(I, K^{-1})-derivation
IIIT, IVT	(K^{-1}, I)-derivation	(K, I)-derivation
IIIST, IVST	(I, K^{-1})-derivation	(I, K)-derivation

Summary

In this talk, we recalled the notion of a tridiagonal pair, and used it to motivate the algebra $\square_q$.

We introduced an infinite-dimensional $\square_q$-module, said to be NIL.

We described the NIL $\square_q$-module from sixteen points of view.

In this description we made use of the free algebra $\mathbb{V}$ on two generators A, B as well as a q-shuffle algebra structure on $\mathbb{V}$.

THANK YOU FOR YOUR ATTENTION!