## 8th PhD Summer School in DM

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# Recent developments on the Spectral Determination of Signed Graphs

# Outline

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# What is Spectral Graph Theory?

Spectral Graph Theory is a mathematical discipline which studies graphs via Linear Algebra (or Matrix Theory).

Main feature: Graph Spectrum = Graph Invariant

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Spectral Graph Theory is a mathematical discipline which studies graphs via Linear Algebra (or Matrix Theory).

Main feature: Graph Spectrum = Graph Invariant

- So we can say that we study unlabelled graphs (or graphs determined up to isomorphism).
- Unfortunately, graph structure is not completely encapsulated into its spectrum (of various matrices).

# The adjacency matrix

The adjacency matrix (or $A$-matrix) of a simple graph G is the $n \times n$ matrix $A = A(G) = (a_{ij})$, where:

$$a_{ij} = \begin{cases} 1 & \text{se } v_i \sim v_j; \\ 0 & \text{se } v_i \nsim v_j \end{cases} .$$

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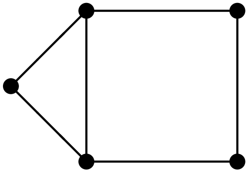

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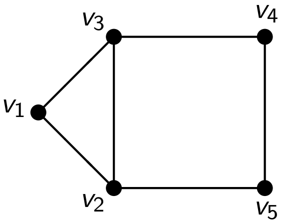

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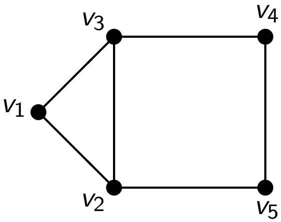
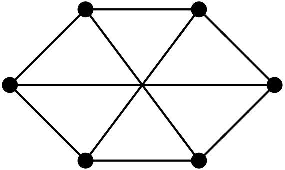



$$\rightsquigarrow \qquad A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Spectral Graph Theory = Combinatorial Theory of $\{0, 1\}$-matrices.

# The adjacency matrix

One more example:



$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Spectrum $\{-3, 0, 0, 0, 0, 3\}$

# The adjacency matrix

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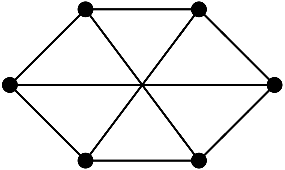

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$$\text{Spectrum } \{-3, 0, 0, 0, 0, 3\}$$

Several combinatorial properties of the graph can be deduced from the algebraic invariants of the adjacency matrix.

A well-known result:

**Theorem (Coulson, Rushbrooke 1940, Sachs 1966)**

*A graph is bipartite if and only if the spectrum of the adjacency matrix is symmetric w.r.t.* 0*.*

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*A graph is bipartite if and only if the spectrum of the adjacency matrix is symmetric w.r.t.* $0$.

We can also deduce if the graph is $k$-regular.

**Proposition**

*Let* $A$ *be the adjacency matrix of a graphs* $G$. *The* $\{i,j\}$*-entry of* $A^p$ *counts the number of walks of length* $p$ *from* $i$ *to* $j$.

**Corollary**

*The trace of* $A^p$ *counts the number of closed walks of length* $p$.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the adjacency eigenvalues of a graph $G$.

**Proposition**

*The order of $G$ is $n$ and the size of $G$ is $\frac{1}{2} \sum \lambda_i^2$.*

**Theorem**

*A graph is regular if and only if $\lambda_1$ equals the average degree $\frac{m}{n}$.*

If the graph is $k$-regular, $A\mathbf{j} = k\mathbf{j}$, where $\mathbf{j}$ is the all-one vector.
(row sums are constant)

# Structural properties deduced from the $A$-spectrum

From the $A$-eigenvalues of $G$ we get

- number of vertice and edges;
- $G$ being bipartite or not;
- $G$ being $k$-regular or not;
- number of triangles ($\frac{1}{6}\sum \lambda_i^3$);
- number of closed walks of length $p$ ($\sum \lambda_i^p$).

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- number of triangles ($\frac{1}{6}\sum\lambda_i^3$);
- number of closed walks of length $p$ ($\sum\lambda_i^p$).

On the other hand we do not get

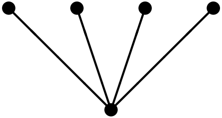
- whether $G$ is connected or not;

- the degree sequence;

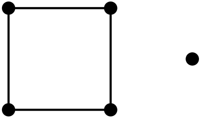
- the number of quadrangles;

- when $G$ bipartite, the size of the classes.

# The most abused example in SGT



$K_{1,4}$



$C_4 \cup K_1$

The $A$-spectrum does not determine:

- if the graph is connected;
- if the graph is acyclic;
- the degree sequence;
- the size of the color classes;
- the number of quadrangles.

# Other important Graph Matrices

Graph matrices with significant structural properties embedded in the spectrum.

- The Laplacian matrix $L = D - A$;
- The signless Laplacian $Q = D + A$;
- The Seidel matrix $S = J - I - 2A$;
- The Generalized matrix $M = \alpha A + \beta D + \gamma J + \delta I$, with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$;
- others (not considered here).

Both $L$ and $Q$ are symmetric diagonally dominant real matrix with nonnegative diagonal entries, hence they are both *positive semidefinite*.

We briefly mention some of their properties.

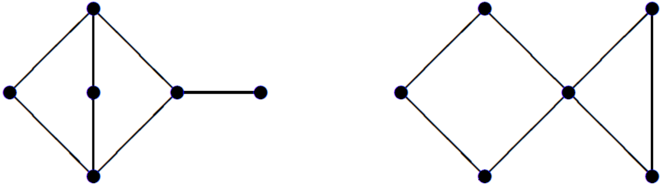
# Structural properties embedded in the spectrum of $L$

For the Laplacian matrix $L = D - A$:

- $L$ is always singular, the multiplicity of the 0-eigenvalue counts the number of connected components. $L\mathbf{j} = \mathbf{0}$;
- The number of vertices is $n$, the number of edges equals $\frac{1}{2}\sum \lambda_i$;
- The number of spanning forest on given number of components (Matrix-tree Theorem);
- Whether $G$ is regular or not;
- The quantities $\sum d_i^2$ and $\sum d_i^3 + 6t$, where the $d_i$'s are degree and $t$ counts triangles in $G$.

Also, If $G$ is connected, the least positive Laplacian eigenvalue (say $\lambda_{n-1}$) is known as *algebraic connectivity*, it measures how far is the graph from being disconnected.

# Smallest $L$-cospectral pairs



The $L$-spectrum does not determine:

- if the graph is bipartite;

- the degree sequence;

- the number of triangles.

# Structural properties embedded in the spectrum of $Q$

For the signless Laplacian matrix $Q = D + A$:

- $Q$ is singular if some component is bipartite, the multiplicity of the 0-eigenvalue counts the number of bipartite components;

- The number of vertices is $n$, the number of edges equals $\sum \lambda_i$;

- The number of $n$-subgraphs whose components are trees or unicyclic graphs (Matrix-tree Theorem);

- Whether $G$ is regular or not;

- The quantities $\sum d_i^2$ and $\sum d_i^3 + 6t$, where the $d_i$'s are degree and $t$ counts triangles in $G$.

- $Q^k$ counts the number of *semi-edge walks* (or, *lazy walks*).

If $G$ is connected, the least eigenvalue (say $\lambda_n$) is known as *algebraic bipartiteness*.

# Smallest $Q$-cospectral pair

The $Q$-spectrum does not determine:

- if the graph is connected;

- if the graph is bipartite;

- the degree sequence;

- the number of triangles.

# Some links between various spectra

- $L$-spectrum and $Q$-spectrum coincide for bipartite graphs.
- We also have links between $A$-spectrum and the $Q$-spectrum:

$$\begin{aligned}\phi(A_{\mathcal{L}(G)}, x) &= (x+2)^{m-n}\phi(Q_G, x+2),\\ \phi(A_{\mathcal{S}(G)}, x) &= x^{m-n}\phi(Q_G, x^2).\end{aligned}$$

$\mathcal{L}(G)$ and $\mathcal{S}(G)$ are the line and subdivision graphs of $G$, resp.

- There is also a formula linking the $L$, $Q$-spectra:

$$\phi(Q_{G\times K_2}, x) = \phi(Q_G, x)\phi(L_G, x) = \phi(L_{G\times K_2}, x)$$

So many results from one theory can be (without proofs) formulated in other theories.

# An important question

QUESTION: To what extent is the graph determined by its $M$-spectrum?

ANSWER: Several structural properties of graphs are not embedded in their spectra. However, it is possible that almost any graph has his own unique $M$-spectrum.

In fact, Haemers has conjectured that almost all graphs are determined by their adjacency spectrum.

# Our main problem

DETERMINATION PROBLEM: to identify graph up to isomorphism from its spectrum, and possibly some further spectral invariants.

CHARACTERIZATION PROBLEM: to characterize graph from its spectrum (say within some class); an example: strongly regular graphs; graph with "small" spectral radius.

The second problem is more general then the first one.

# Research in Spectral Characterization Problems

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- Methods leading to the construction of cospectral trees/graphs;
- Spectral determination problems for given classes of graphs;
- Investigations studying the conditions for which Graphs are Determined by the Spectrum.

## Construction of Cospectral Graphs

in the '70s of the past century, Schwenk observed from the formula decomposing the characteristic polynomial (Schwenk's formula for vertices) the following fact:

**Proposition**

*Let $T$ be a tree and $u$ and $v$ two vertices such that $T-u$ and $T-v$ are isomorphic. If $T_w$ denotes the tree $T$ rooted at $w$*

$$\phi(T_u \cdot T') = \phi(T_v \cdot T').$$

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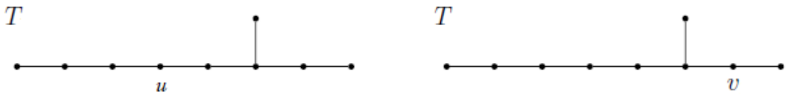
$$\phi(T_u \cdot T') = \phi(T_v \cdot T').$$

In fact, the assumption of $T-u$ and $T-v$ being isomorphic can be weakened by the condition $\phi(T-u) = \phi(T-v)$:



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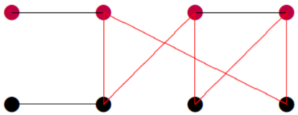
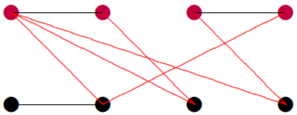
Godsil and McKay invented the machine to construct cospectral graphs with cospectral complements.





# Godsil-McKay Switching

**Theorem**

*Let $G$ be a graph and let $\{X_1, ..., X_t, Y\}$ be a partition of the vertex set $V(G)$ of $G$. Suppose that for every vertex $x \in Y$ and every $i \in \{1, \ldots, t\}$, $x$ has either 0, $\frac{1}{2}|X_i|$ or $|Xi|$ neighbors in $Xi$. Moreover, suppose that for all $i, j \in 1, ..., t$ every vertex $x \in X_i$ has the same number of neighbors in $X_j$. Make a new graph $G$ as follows. For each $x \in Y$ and $i \in \{1, ..., t\}$ such that $x$ has $\frac{1}{2}|X_i|$ neighbors in $X_i$ switch the edges with the non-edges. Then $G$ and $G$ are cospectral.*

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Observe that the above procedure can produce just isomorphic graphs!

# Godsil-McKay Switching - variant

This variant can be adapted to various graph matrices...

**Theorem**

*Let $N$ be a $(0,1)$-matrix of size $b \times c$ (say) whose column sums are $0$, $b$ or $b/2$. Define $\tilde{N}$ to be the matrix obtained from $N$ by replacing each column $\mathbf{v}$ with $b/2$ ones by its complement $\mathbf{j} - \mathbf{v}$. Let $B$ be a symmetric $b \times b$ matrix with constant row (and column) sums, and let $C$ be a symmetric $c \times c$ matrix. Put*

$$M = \begin{pmatrix} B & N \\ N^T & C \end{pmatrix} \quad \text{and} \quad \tilde{M} = \begin{pmatrix} B & \tilde{N} \\ \tilde{N}^T & C \end{pmatrix}$$

*Then $M$ and $\tilde{M}$ are cospectral.*

$M$ can be the $A$, $-L$, $Q$, and others...

# Wang and Xu method

Find all rational orthogonal matrices $Q$ such that $Q^T AQ$ is a $(0,1)$-matrix.

If each $Q$ is a permutation matrix, then the graph is determined by the adjacency spectrum.

- (Wang 2014) Graphs for which $|W|/2^{\lfloor n/2 \rfloor}$ is odd and square free are determined by the generalized spectrum ($W$ is the walk matrix).

- (ORourke, Touri 2016) Almost all graphs are controllable (the walk matrix is nonsingular).

In the next slides we move to signed graphs. Signed graphs are graphs whose edges get a sign (either "$+$" or "$-$"). Signs are usually interpreted with the values $+1$ or $-1$.

# Outline

The adjacency Spectral Theory of Graphs can be considered the Combinatorial matrix theory of symmetric $\{0, 1\}$-matrices.

We now *switch* to signed graphs whose adjacency spectral theory can be considered as the Combinatorial matrix theory of symmetric $\{0, 1, -1\}$ matrices...

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We now *switch* to signed graphs whose adjacency spectral theory can be considered as the Combinatorial matrix theory of symmetric $\{0, 1, -1\}$ matrices...

We will see how these theories are related. A few more definitions ahead.

# Signed Graphs

A *signed graph* $\Gamma$ is an ordered pair $(G, \sigma) = G_\sigma$, where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+, -\}$ is the signature function (or sign mapping) on the edges of $G$.

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In general, the underlying graph $G$ may have loops, multiple edges, half-edges (with only one endpoint), and loose edges (with no endpoints). However, half and loose edges do not receive signs. If $C$ is a cycle in $\Gamma$, the sign of the $C$ is the product of its edges signs.

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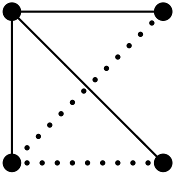

Example of a signed graph.

Positive edges are bold lines, negative edges are dotted lines.

# More on Signed Graphs

The signs of the cycles have a central role in the theory of signed graphs.

The theory of signed graphs is related to the theory of *Biased Graphs* rather than *Weighted Graphs*. If all cycles are positive, the signed graph is "equivalent" to the underlying graph.

**Definition**

*A signed graph is said to be balanced if and only if any of its cycles contains an even number of negative edges.*

# Balance

The first characterization of balance is due to Harary:

**Theorem (Harary, 1953)**

*A signed graph is balanced iff its vertex set can be divided into two sets (either of which may be empty), so that each edge between the sets is negative and each edge within a set is positive.*

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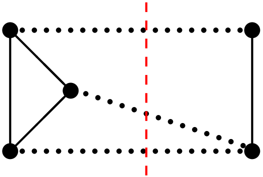
*A signed graph is balanced iff its vertex set can be divided into two sets (either of which may be empty), so that each edge between the sets is negative and each edge within a set is positive.*

In fact, being balanced is a generalization of bipartiteness in (unsigned) graphs.



A balanced signed graph. The dashed line separates the two clusters.

# Signature Switching

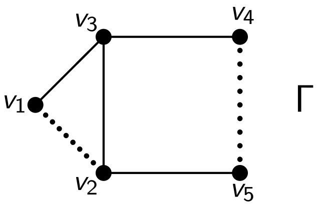
**Definition**

*Let $\Gamma = (G, \sigma)$ be a signed graph and $U \subseteq V(G)$. The signed graph $\Gamma^U$ obtained by reversing the edges signs in the cut $[U; U^c]$ is a* (sign) switching *of $\Gamma$. We also say that the signatures of $\Gamma^U$ and $\Gamma$ are* equivalent.

The signature switching preserves the set of the positive cycles.

In general, we say that two signed graphs are *switching isomorphic* if their underlying graphs are isomorphic and the signatures are switching equivalent. The set of signed graphs switching isomorphic to $\Gamma$ is the *switching isomorphism class* of $\Gamma$, written $[\Gamma]$.

# Example of switching equivalent graphs



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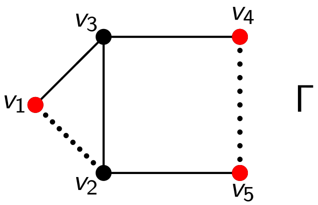


Let $U = \{v_1, v_4, v_5\}$.

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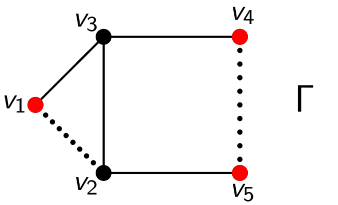


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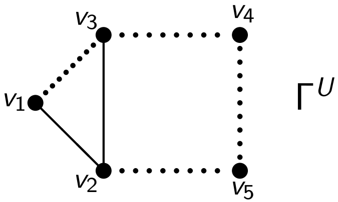



Note that the switching preserves the sign of the cycles!

# Balance and switching

The following theorem is due to Harary:

**Theorem**

*A signed graph is balanced if and only if it is switching equivalent to the the all positive signature.*

In fact, (unsigned) graphs can be considered as balanced signed graphs.
A signed graph that is switching equivalent to the all negative signature is said to be *antibalanced*.

# Matrices of Signed Graphs

Of course, matrices can be associated to signed graphs.

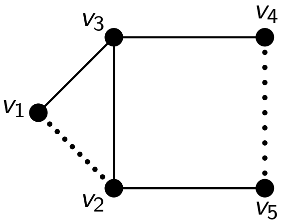
- The (signed) adjacency matrix $A(G_\sigma)$. It is defined as the unsigned one, but negative edges get $-1$;
- The Laplacian matrix $L(G_\sigma) = D(G) - A(G_\sigma)$, where $D$ is the diagonal matrix of $G$.

Note: The Seidel matrix of a simple graph is the special case of the Adjacency matrix of a signed complete graph.

# Adjacency matrix of Signed Graphs

The adjacency matrix is defined as $A(\Gamma) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i v_j), & \text{if } v_i \sim v_j; \\ 0, & \text{if } v_i \not\sim v_j. \end{cases}$$

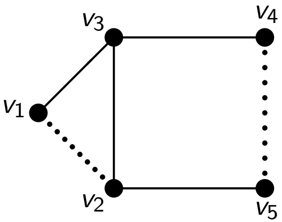



$$\rightsquigarrow \quad A(\Gamma) = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

# Laplacian of Signed Graphs

The Laplacian matrix of $\Gamma = (G, \sigma)$ is defined as
$L(\Gamma) = D(G) - A(\Gamma) = (l_{ij})$

$$l_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j; \\ -\sigma(v_i v_j), & \text{if } i \neq j. \end{cases}$$




$$\rightsquigarrow \quad L(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{pmatrix}$$

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What happens if we pick two signed graphs from the same switching class?

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Switching has a matrix counterpart. In fact, Let $\Gamma$ and $\Gamma' = \Gamma^U$ be two switching equivalent graphs.
Consider the matrix $S_U = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ such that

$$s_i = \begin{cases} +1, & i \in U \\ -1, & i \in \Gamma \setminus U \end{cases}$$

$S_U$ is called a *signature matrix* (or *state* matrix).

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Switching has a matrix counterpart. In fact, Let $\Gamma$ and $\Gamma' = \Gamma^U$ be two switching equivalent graphs.
Consider the matrix $S_U = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ such that

$$s_i = \begin{cases} +1, & i \in U \\ -1, & i \in \Gamma \setminus U \end{cases}$$

$S_U$ is called a *signature matrix* (or *state* matrix).

It is easy to check that

$$A(\Gamma^U) = S_U \ A(\Gamma) \ S_U \quad \text{and} \quad L(\Gamma^U) = S_U \ L(\Gamma) \ S_U.$$

Hence, signed graphs from the same switching class are cospectral.

# More on switching

The following theorem holds:

**Theorem**

*Two signed graphs with the same underlying graph are switching equivalent if and only if they have the same class of positive cycles.*

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*Two signed graphs with the same underlying graph are switching equivalent if and only if they have the same class of positive cycles.*

Switching does not change the eigenvalues but it does change the eigenspaces:

**Proposition**

*If a signed graph is switched w.r.t. a subset of vertices, then eigenvector components related to such vertices are changed in signs.*

# Switching Isomorphism

While the spectrum of (unsigned) graphs does not distinguish isomorphic graphs, the spectrum of signed graphs does not distinguish switching equivalent graphs.

**Definition**

*Two signed graphs $G_\sigma$ and $G'_{\sigma'}$ are said to be swtiching isomorphich if and only if there is an isomorphism of underlying graphs that preserves the signs of cycles.*

In matrix form, $G_\sigma$ and $G'_{\sigma'}$ are switching isomorphic if and only if there exist a permutation matrix $P$ and a state matrix $S$ such that

$$A(G_\sigma) = (PS)^{-1}\, A(G'_{\sigma'})\, (PS).$$

# Spectral determination problems

Of course, the spectral determination problem can be re-stated for signed graphs. It seems that it was first stated by Acharya.

**Definition**

*A signed graph $G_\sigma$ is said to be determined by the spectrum of the matrix $M(G_\sigma)$ (in short, M-DS) iff any cospectral signed graphs to $G_\sigma$ is switching isomorphic as well.*

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Note that the spectral determination problem is much more difficult w.r.t. the unsigned graph counterpart. In fact, we have more possibility of matrix similarity and the spectra of signed graphs, unfortunately, bind *weaker* structural properties.

# Structural properties deduced from the $A$-spectrum

From the $A$-eigenvalues of $G_\sigma$ we get

- number of vertice and edges;
- difference between the number of positive and negative triangles ($\frac{1}{6}\sum \lambda_i^3$);
- difference between the number of positive and negative closed walks of length $p$ ($\sum \lambda_i^p$).

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One important feature of the adjacency spectrum of (unsigned) graphs has not been mentioned yet.

# One more interesting problem

One of the most well-known results in the adjacency spectral theory of (unsigned) graphs is the following:

**Theorem**

*A graph is bipartite if and only if its adjacency spectrum is symmetric with respect to the origin.*

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*Let $G$ be a graph, $\rho$ be the spectral radius and $\lambda_n$ be the least eigenvalue. Then $\rho(G) = -\lambda_n$ if and only if $G$ is bipartite.*

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QUESTION: Does something similar hold for signed graphs?

# Sign-symmetric signed graphs

A signed graph $\Gamma = (G, \sigma)$ is said to be sign-symmetric if $\Gamma$ is isomorphic to $-\Gamma = (G, -\sigma)$.

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A signed graph $\Gamma = (G, \sigma)$ is said to be sign-symmetric if $\Gamma$ is isomorphic to $-\Gamma = (G, -\sigma)$.

It is not difficult to observe that the signature-reversal changes the sign of odd cycles and leaves unaffected the sign of even cycles.

Since bipartite (unsigned) graphs are odd-cycle free, it happens that bipartite graphs are a special case of sign-symmetric signed graphs.

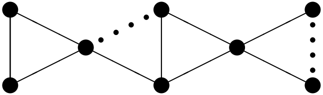

Figure: An example of sign-symmetric signed graph.

# Two conjectures

On the other hand, if $\Gamma$ is isomorphic to $-\Gamma$, then $A$ and $-A$ are similar and we get easily get:

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*A signed graph $\Gamma$ is sign-symmetric if and only if its adjacency spectrum is symmetric with respect to the origin.*

**Conjecture**

*Let $\Gamma$ be a signed graph, $\rho$ be the spectral radius, $\lambda_1$ and $\lambda_n$ be the largest and the least eigenvalue, respectively.*
*Then $\rho(G) = -\lambda_n = \lambda_1$ if and only if $\Gamma$ is sign-symmetric.*

# Coefficient Theorem for the adjacency polynomial

The elementary figures are the graphs $K_2$ and $C_n$; a basic figure is the disjoint union of elementary figures. Let $\mathcal{B}_i$ be the set of basic figures on $i$ vertices, $p(B)$ # components of $B$, $|c(B)|$ # of cycles in $B$, and $\sigma(B) = \prod_{C \in c(B)} \sigma(C)$.

**Theorem**

*Let $\Gamma$ be a signed graph and let $\phi(\Gamma, x) = \sum_{i=0}^{n} a_i x^{n-1}$ be its adjacency characteristic polynomial. Then*

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} 2^{|c(B)|} \sigma(B),$$

# Further facts on the $A$-spectrum

Spectral criterion for the signed graph being balanced, due to Acharia:

**Theorem**

*Let $G_\sigma$ be a signed graph. Then, $G_\sigma$ is balanced if and only if $G_\sigma$ is $A$-cospectral to $G$.*

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Detecting if $G_\sigma$ is regular is somewhat lost with signed graphs:

**Theorem**

*$G$ is regular if and only if $\mathbf{j}$ is an eigenvector of both $A(G_\sigma)$ and $A(G)$.*

Observe that if $\mathbf{j}$ is an eigenvector of $A(G_\sigma)$, then the signed degree $sd(v) = d^+(v) - d^-(v)$ is constant.

# Interlacing theorem (vertex variant)

Interlacing theorem holds for the adjacency matrix of signed graphs:

**Theorem**

*Let $\Gamma = (G, \sigma)$ be a signed graph and $\Gamma - v$ be the signed graph obtained from $\Gamma$ by deleting the vertex $v$. If $\lambda_i$'s are the adjacency eigenvalues, then*

$$\lambda_1(\Gamma) \geq \lambda_1(\Gamma - v) \geq \lambda_2(\Gamma) \geq \lambda_2(\Gamma - v) \geq \cdots \geq \lambda_{n-1}(\Gamma - v) \geq \lambda_n(\Gamma).$$

# Laplacian Spectral Moments

Let $T_k = \sum_{i=1}^{k} \mu_i^k$ $(k = 0, 1, 2, \ldots)$ be the $k$-th spectral moment for the Laplacian spectrum of a signed graph $\Gamma$.

**Theorem**

*Let $\Gamma = (G, \sigma)$ be a signed graph with $n$ vertices, $m$ edges, $t^+$ balanced triangles, $t^-$ unbalanced triangles, and vertex degrees $d_1, d_2, \ldots, d_n$. We have*

$$T_1 = \sum_{i=1}^{n} d_i, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6(t^- - t^+) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

# Structural properties deduced from the $L$-spectrum

For the Laplacian matrix $L$:

- The multiplicity of the 0-eigenvalue counts the number of balanced components;
- The number of vertices is $n$, the number of edges equals $\frac{1}{2}\sum \lambda_i$;
- The number of subgraphs on $k$ vertices whose components are trees or unbalanced unicyclic graphs (Matrix-tree Theorem);
- The quantity $\sum d_i^2$;
- The quantity $\sum d_i^3 + 6(t^- - t^+)$.

It seems that the $L$-spectrum keeps more structural information about the signed graph.

# The signed Matrix-tree theorem

A signed $TU$-graph is a graph whose components are trees or unbalanced unicyclic graphs (the unique cycle has a negative sign). If $H = T_1 \cup \cdots \cup T_r \cup U_1 \cup \cdots \cup U_s$, then $\gamma(H) = 4^s \prod_{i=1}^{r} |T_i|$.

**Theorem**

*Let $\Gamma$ be a signed graph and $\psi(L(\Gamma), x) = \sum_{i=0}^{n} b_i x^{n-i}$ be its Laplacian polynomial. Then we have:*

$$b_i = (-1)^i \sum_{H \in \mathcal{H}_i} \gamma(H),$$

*where $\mathcal{H}_i$ is the set number of signed $TU$-graphs on $i$ edges.*

# The Interlacing Theorem (edge variant)

**Theorem**

*Let $\Gamma = (G, \sigma)$ be a signed graph and $\Gamma - e$ be the signed graph obtained from $\Gamma$ by deleting the edge $e$. If $\mu_i$'s are the Laplacian eigenvalues, then*

$$\mu_1(\Gamma) \geq \mu_1(\Gamma - e) \geq \mu_2(\Gamma) \geq \mu_2(\Gamma - e) \geq \cdots \geq \mu_n(\Gamma) \geq \mu_n(\Gamma - e).$$

# Outline

# Determination Problems studied so far

Some spectral determination problems already investigated:

- Adjacency Spectral determination of signed paths (S. Akbari, W.H. Haemers, H.R. Maimani, L.P. Majd).
- Adjacency Spectral determination of signed paths (S. Akbari, FB, E. Dodongeh, M.A. Nematollahi).
- Laplacian Spectral determination of Friendship (Firefly) graphs (FB, P. Petecki, J.F. Wang);
- Laplacian Spectral determination of lollipop graphs (FB, P. Petecki);
- Laplacian Spectral determination of signed infinity graphs - connected case (FB, M. Brunetti).

# A-Spectral determination of signed paths

The path $P_n$ is determined by the spectrum if the adjacency matrix. Surprisingly, it is not true when considering signed graphs.

A tentative cospectral mate of a path must have symmetric spectrum, simple eigenvalues and the spectrum must be included in the interval $(-2, 2)$.

Which are the signed graphs whose $A$-spectrum lies in the interval $[2, -2]$?

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Which are the signed graphs whose $A$-spectrum lies in the interval $[2, -2]$?

McKee and Smyth gave the answer to this SCP in 2007.

# Signed graphs with $A$-spectrum in $[-2, 2]$

A cyclotomic graph is a graph whose spectrum is in the interval $[2, -2]$

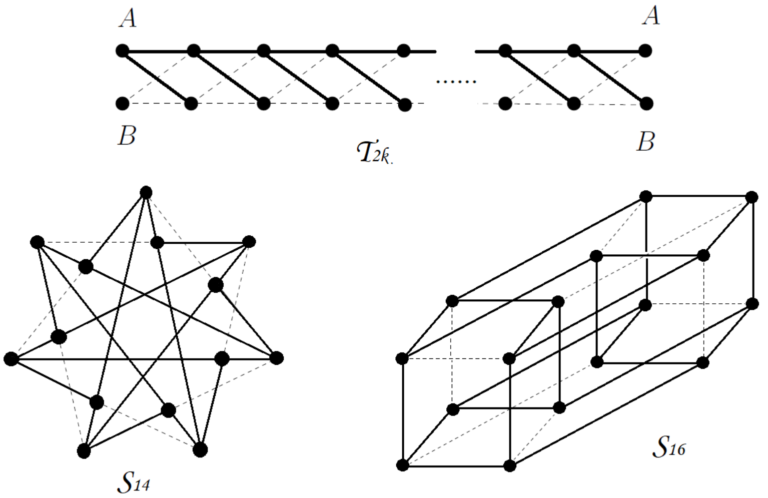
**Theorem (McKee and Smyth, 2007)**

*Every maximal connected cyclotomic signed graph is equivalent to one of the following:*

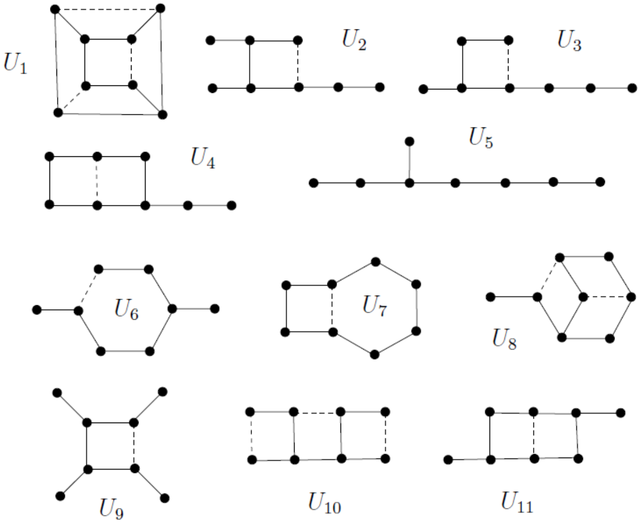
- *For some $k = 3, 4, \ldots$, the 2k-vertex toral tesselation $T_{2k}$;*
- *The 14-vertex signed graph $S_{14}$;*
- *The 16-vertex signed hypercube $S_{16}$.*

*Further, every connected cyclotomic signed graph is contained in a maximal one.*

# Maximal Cyclotomic Signed Graphs



# Signed Graphs with $A$-spectrum in $(-2, 2)$



# Adjacency Spectral Determination of signed paths

**Theorem**

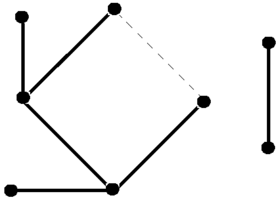
*Let $P_n$ be the path on $n$ vertices. Then $P_n$ is A-DS if and only if*

(a) $n$ *is even and* $n \neq 8, 14$;

(b) $n \equiv 1 (\mathrm{mod}\ 4)$ *and* $n \notin \{13, 17, 29\}$;

(c) $n \equiv 3 (\mathrm{mod}\ 4)$ *and* $n = 3$.

$P_8$



# What about the Laplacian spectral determination of paths

For the Laplacian the determination problem is much easier, since:

- The path is balanced, hence the tentative cospectral mate must have exactly one balanced component;

- The Laplacian spectrum of the path is in the interval $[0, 4)$ and it consists of simple eigenvalues;

- Vertices of degree 3 lead to a spectral radius $\geq 4$;

- Cycles have eigenvalues of multiplicity 2.

Therefore the cospectral mate of a path can be only another path, and the result follows immediately.

## What about the $A, L$-spectral determination of cycles

The adjacency and Laplacian spectral determination of cycles has been studied recently. Cycles are DS as unsigned graphs, but they get several cospectral mates with signed graphs.

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The adjacency and Laplacian spectral determination of cycles has been studied recently. Cycles are DS as unsigned graphs, but they get several cospectral mates with signed graphs.

**Theorem**

*Let $C_n^+$ (resp., $C_n^-$) be the balanced (resp., unbalanced) cycle of order $n = 2^r(2s+1)$. Then the following equality holds:*

$$\begin{aligned} Spec_L(C_n, +) = \; & Spec_L(C_{2^{r-1}(2s+1)}, +) \cup Spec_L(C_{2^{r-1}(2s+1)}, -) \cup \\ & Spec_L(C_{2^t(2s+1)}, +) \bigcup_{i=t}^{r-1} Spec_L(C_{2^i(2s+1)}, -). \end{aligned}$$

W.r.t. Laplacian spectrum, the above is the unique cospectral mate of cycles.

# Admissible signed graphs

Since the cycles get 2 as a simple eigenvalue, the first step was to characterize those signed graphs with such a property. They are called *admissible graphs*.

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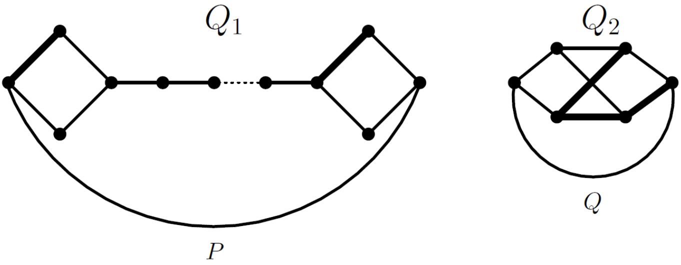
**Theorem**

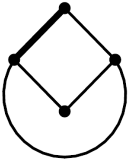
*Let $\Gamma$ be a signed subgraph of $T_{2k}$, then $\Gamma$ is admissible if and only if $\Gamma$ is a proper induced subgraph of $Q_1$ or $Q_2$.*



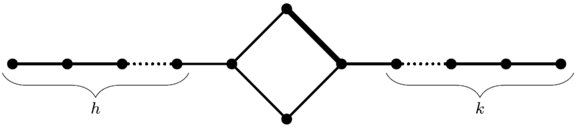
# Adjacency Spectral Determination of Signed Cycles

## Theorem

*Let $(C_n, \sigma)$ be a signed cycle, $n > 4$. Then signed cycles are determined by their adjacency spectrum if and only if $n$ is odd.*




The signed graph $\tilde{Q}_n$.




The signed graph $Q_{h,k}$.

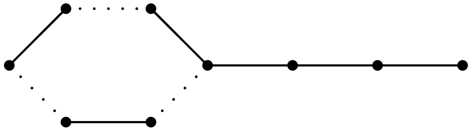
$$(C_{2n}, -) \equiv Q_{n-2,n-2}, \quad (C_{2n}, +) \equiv \tilde{Q}_{n+1} \cup P_{n-1}.$$

# Laplacian Spectral Determination of Lollipops

Lollipop graphs are the coalescence of a cycle and a path. The Laplacian case was studied by Haemers et al., the adjacency case was studied by Boulet and Jouve (a paper of around 50 pages!).

**Theorem (FB, Petecki, 2015)**

*Let $\Gamma$ be a signed lollipop, consisting of a signed cycle having a pendant path. Then the signed lollipops are determined by the spectrum of the Laplacian matrix.*



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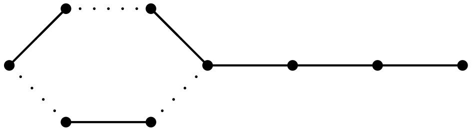

The adjacency case of lollipops has not been studied yet.

# Laplacian Spectral Determination of $\infty$-graphs

Iranmanesh and Saheli, and (independently) Brunetti and myself have found the following table of connected $L$-cospectral mates:

| $\infty$-graphs | Bicyclic cospectral mates | |
|---|---|---|
| $C_{3,4}^{+,+}$ | $\Theta(3)_{4,4,1}^{+,+}$ | (He, van Dam 2018) |
| $C_{3,5}^{+,+}$ | $\Theta(1)_{4,4,1}^{+,+}$ | (He, van Dam 2018) |
| $C_{2k-1,2k}^{-,+}$ | $\Theta(1)_{2k-1,3,2k-2}^{-,-}$ | (Wang, Belardo 2010) |
| $C_{2k,2k+1}^{+,-}$ | $\Theta(1)_{2k,3,2k-1}^{+,-}$ | (Wang, Belardo 2010) |
| $C_{2k-1,2k}^{+,-}$ | $\Theta(1)_{2k-1,3,2k-2}^{+,-}$ | |
| $C_{2k,2k+1}^{-,+}$ | $\Theta(1)_{2k,3,2k-1}^{-,-}$ | |

# Open Problems

The following conjecture comes from obtained results, it needs a computational validation:

**Conjecture**

*If we consider signed graphs, the Laplacian spectrum distinguishes non-(switching)isomorphic signed graphs better than the adjacency spectrum.*

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*If we consider signed graphs, the Laplacian spectrum distinguishes non-(switching)isomorphic signed graphs better than the adjacency spectrum.*

Also:
- The method of Wang and Xu can be adapted to signed graphs?
- It is possible to find routines to built cospectral signed graphs? (Which avoid the restriction on $N$)

# Spectral radius: What we lose with those $-1$'s?

By dealing with signed graphs, we get the following:

- The largest eigenvalue could not be the spectral radius;
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- Adding edges might reduce the largest eigenvalue;
- Powers of the adjacency matrix count the difference of signed walks.

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- Powers of the adjacency matrix count the difference of signed walks.

QUESTION: Is it still relevant to study signed graphs in terms of the spectral radius?

# The spectral radius of signed graphs

Still, two important results do hold:

**Theorem**

*Let $A(\Gamma)$ be the adjacency matrix of $\Gamma = (G, \sigma)$, then*

$$\rho(\Gamma) \leq \rho(G) \leq \Delta(G).$$

**Theorem (Cauchy's Interlacing Theorem)**

*If $\Lambda$ is an induced subgraph of $\Gamma$, then $\rho(\Lambda) \leq \rho(\Gamma)$.*

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**Theorem (Cauchy's Interlacing Theorem)**

*If $\Lambda$ is an induced subgraph of $\Gamma$, then $\rho(\Lambda) \leq \rho(\Gamma)$.*

So it makes sense to characterize the signed graphs according to the magnitude of the spectral radius.

# Hoffmann program for signed graphs

In 1989, Cvektović et al, Brouwer and Neumaier determined all connected graphs whose spectral radius does not exceed $\sqrt{2+\sqrt{5}}$. These graphs are (chemical) trees with at most two vertices of degree 3.

# Hoffmann program for signed graphs

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Limit points studied by Hoffmann for graphs are still valid in the context of signed graphs. Therefore, an interesting problem is to include signed graphs in the above mentioned set. This is also known as Hoffmann program.

**Problem (Hoffmann Program for Signed Graphs)**

*Characterize all connected signed graphs whose spectral radius does not exceed* $\sqrt{2+\sqrt{5}}$.

*Thank you!!*

## 8th PhD Summer School in DM

**Rogla, July 2, 2018**

**Francesco Belardo**

University of Naples "Federico II"

# Recent developments on the Spectral Determination of Signed Graphs