Some (recent) results on half-arc-transitivity of graphs

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Before we focus on half-arc-transitivity...

- $\Gamma = (V, E)$ will always be a (regular) graph,
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  - G can have at most two orbits on V;

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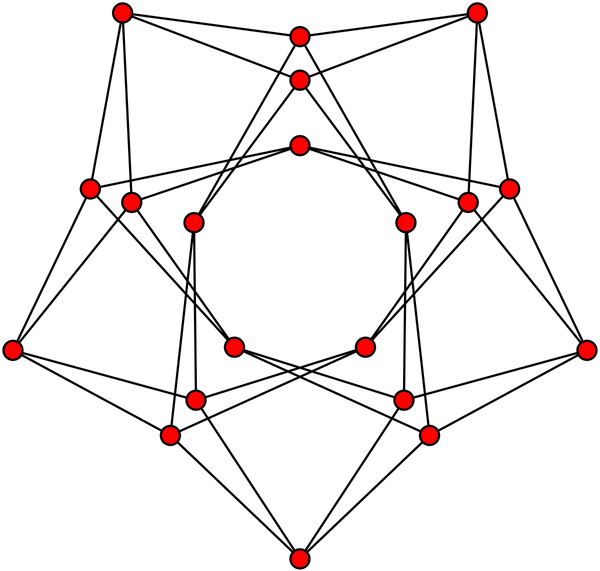
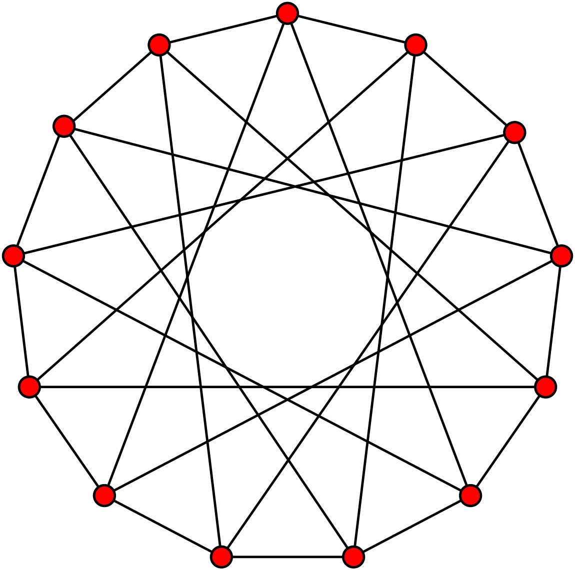
- Do we have all three possibilities?
- For all edge-transitive $G \le \mathrm{Aut}(\Gamma)$?

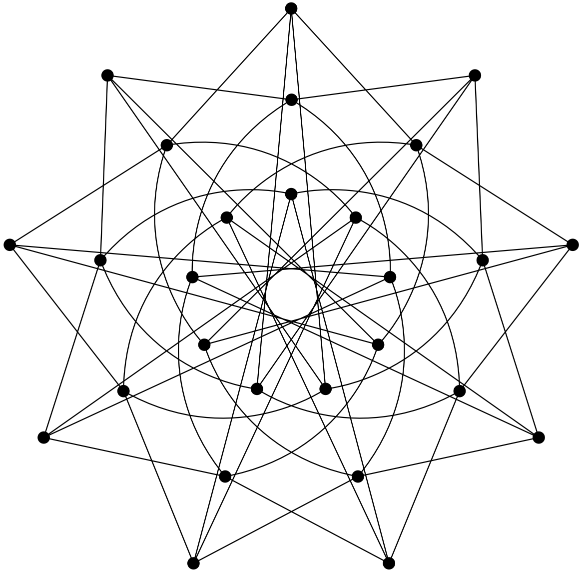
A semi-symmetric graph



An arc-transitive graph



A half-arc-transitive graph



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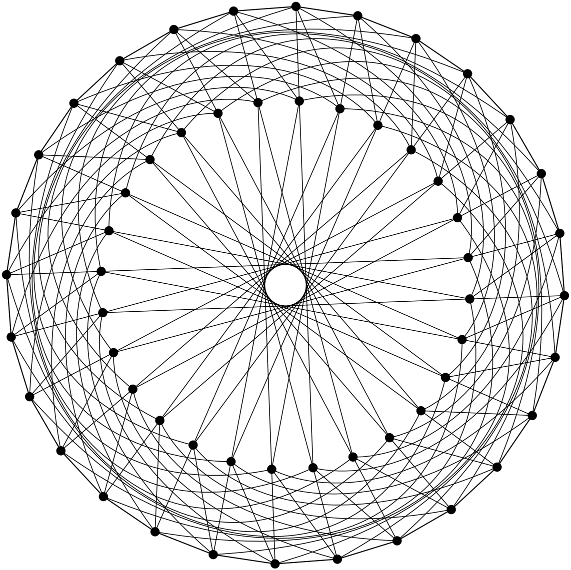

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  - tetravalent HAT graphs with large abelian vertex-stabilizers (Marušič 2005);
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  - construction of a census of all tetravalent HAT graphs up to order 1000 (Potočnik, Spiga, Verret 2015);
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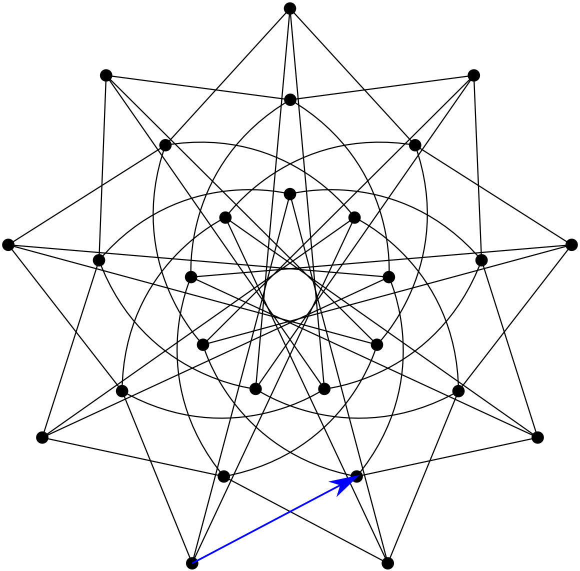
- Gives good insight into the structure of Γ.
- What can be said about intersections of G-alternets? Or is $\mathcal{R}$ be universal?

The reachability relation



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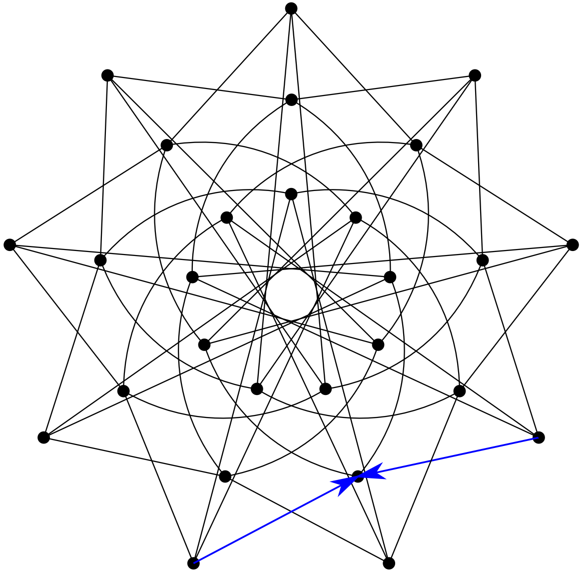

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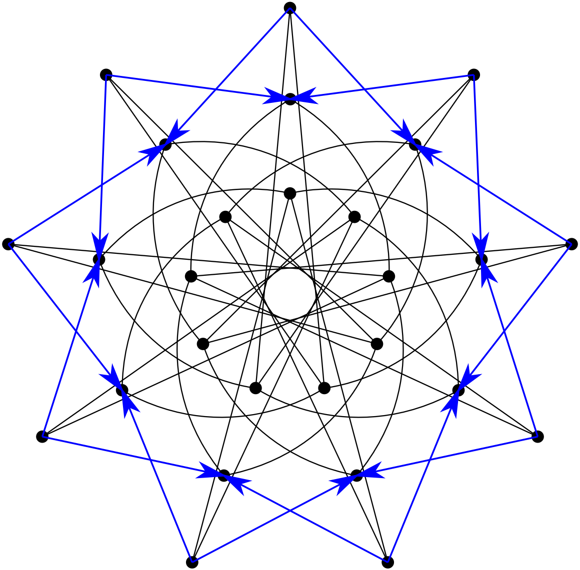

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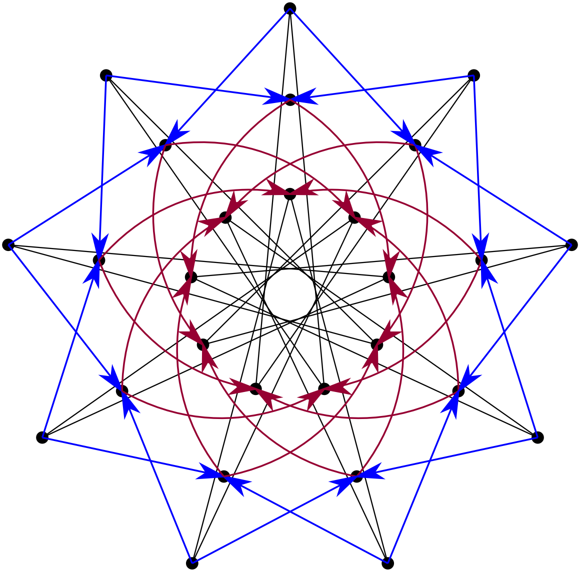

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  - The Bouwer graphs are TA (HAT ones classified by Conder, Žitnik 2016), the generalized Bouwer graphs also (classified by Ramos Rivera, Š 2017).
  - Many more TA examples: wreath products and deleted wreath products (Dobson, Miklavič, Š 2???).

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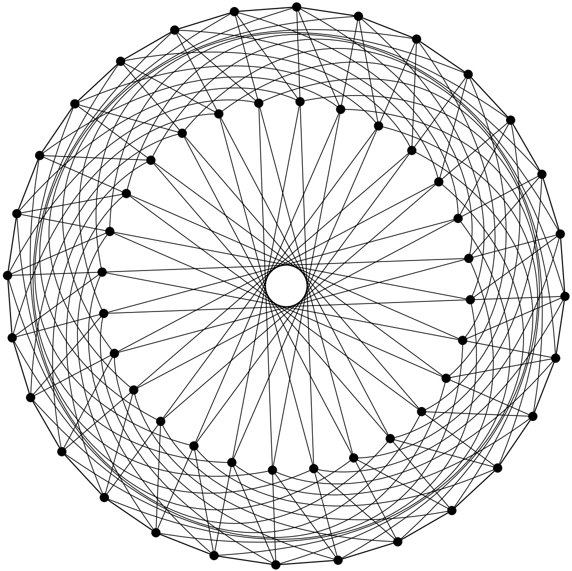
  - Infinite family of imprimitive 6-valent HAT graphs with universal $\mathcal{R}$ (Jajcay, Miklavič, Š, Vasiljević 2???).
  - Infinitely many $6k$-valent HAT graphs with universal $\mathcal{R}$ for each $k \geq 1$ (Dobson, Miklavič, Š 2???).

The smallest example in the infinite family



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- $\mathrm{att}_G(\Gamma)$ divides $2\mathrm{rad}_G(\Gamma)$ (unless $\mathrm{att}_G(\Gamma) = 2\mathrm{rad}_G(\Gamma)$).
- Extremal cases:
  - **tightly G-attached** ($\mathrm{att}_G(\Gamma) = \mathrm{rad}_G(\Gamma)$);
  - **antipodally G-attached** ($\mathrm{att}_G(\Gamma) = 2$);
  - **loosely G-attached (**$\mathrm{att}_G(\Gamma) = 1$**).**

Why is this important?

Theorem (Marušič, Praeger, 1999)

Let Γ be a tetravalent G-HAT graph for some $G \leq \mathrm{Aut}(\Gamma)$ such that $a \neq 2r$, where $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. Let $\mathcal{B}$ be the set of all G-attachment sets (or ...) and let $\Gamma_{\mathcal{B}}$ be the quotient graph of Γ with respect to $\mathcal{B}$. Then one of the following holds:

(i) *$a = r$, that is Γ is tightly G-attached;*

(ii) *$a < r$ and $\Gamma_{\mathcal{B}}$ is a tetravalent graph admitting a HAT action of a quotient group $\bar{G}$ of G and is loosely $\bar{G}$-attached or antipodally $\bar{G}$-attached, depending on whether a divides r or not, respectively.*

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- What is the kernel of the action of G on $\Gamma_{\mathcal{B}}$?
- What can we say about the case when a does not divide r?

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- Census (PSV2015) shows that there are many examples where $\mathrm{att}_G(\Gamma)$ does not divide $\mathrm{rad}_G(\Gamma)$. But, there is no HAT tetravalent graph of order up to 1000 with this property.

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Theorem (Potočnik, Š, 2017)

For every tetravalent HAT graph Γ with $\mathrm{rad}(\Gamma)$ twice an odd number, $\mathrm{att}(\Gamma)$ divides $\mathrm{rad}(\Gamma)$.

The smallest possible "bad" case

- Characterization of the case $\mathrm{rad}_G(\Gamma) = 3$ and $\mathrm{att}_G(\Gamma) = 2$.

Theorem (Potočnik, Š, 2017)

Let Γ be a connected tetravalent graph. Then Γ is G-HAT for some $G \leq \mathrm{Aut}(\Gamma)$ with $\mathrm{rad}_G(\Gamma) = 3$ and $\mathrm{att}_G(\Gamma) = 2$ if and only if Γ is the dart graph of some 2-arc-transitive cubic graph.

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- The key ingredients of the proof:
  - The *dart graph* construction (introduced by Hill and Wilson in 2012).
  - The graph $\mathrm{Alt}_G(\Gamma)$ of G-alternating cycles.

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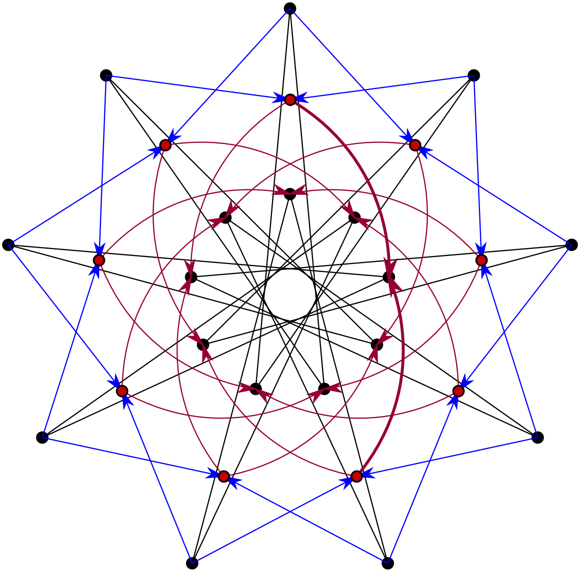
- The **G-alternating jump** of Γ (Ramos Rivera, Š, 2???).

The alternating jump in Doyle-Holt



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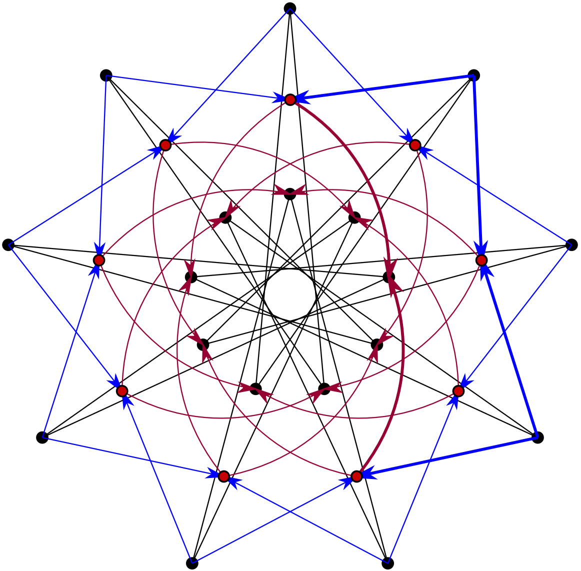

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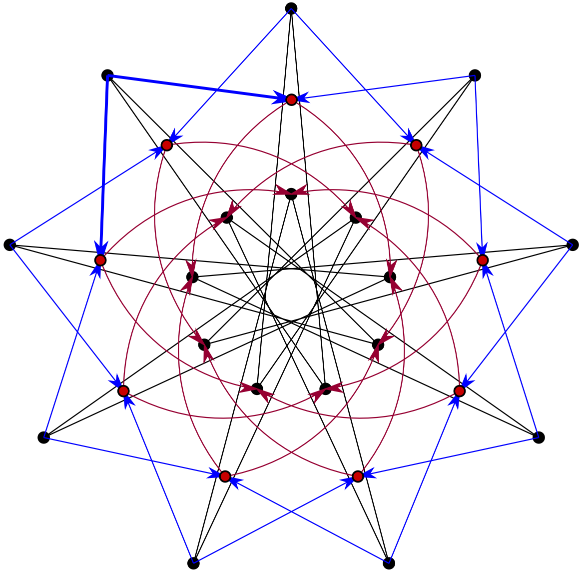

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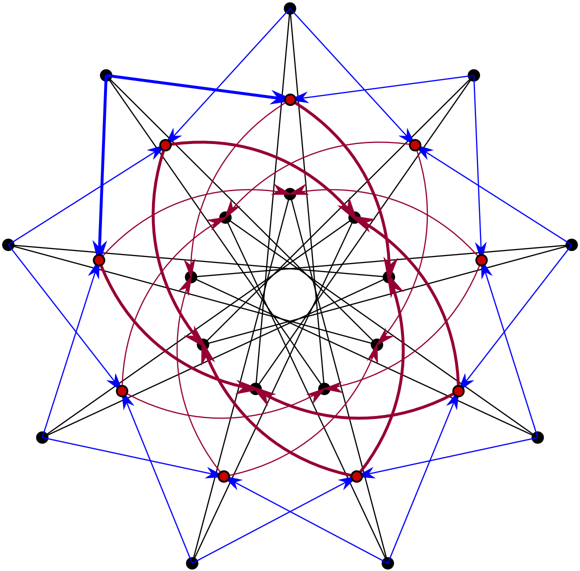
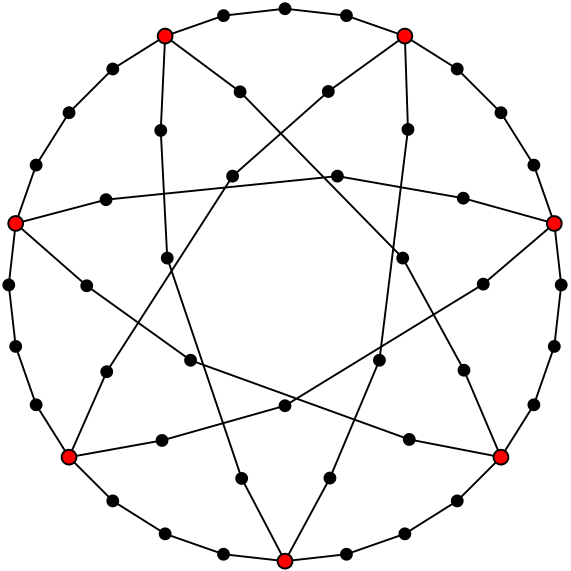


The alternating jump in HAT[84,1]



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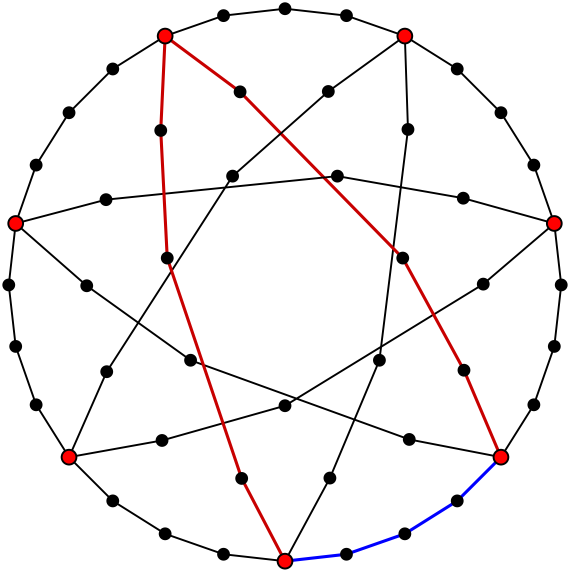

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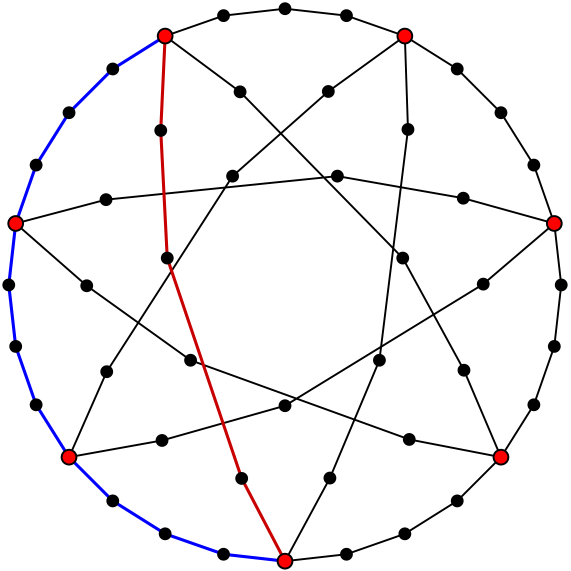

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- Γ a tetravalent G-HAT graph for $G \leq \mathrm{Aut}(\Gamma)$, $r = \mathrm{rad}_G(\Gamma)$, $a = \mathrm{att}_G(\Gamma)$, $\ell = 2r/a$.
- $C = (u_0, u_1, \ldots, u_{2r-1})$ and $C' = (v_0, v_1, \ldots, v_{2r-1})$ the two G-alternating cycles containing $w = u_0 = v_0$.

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- So, $\mathrm{jum}(\Gamma) = 2$ for Doyle-Holt and also for HAT[84,1].

Results (Ramos Rivera, Š, 2???)

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Let Γ, G, a, r, q_t, q_h *and* $q = \mathrm{jum}_G(\Gamma)$ *be as above. Then* $q_t q_h \equiv \pm 1 \pmod a$. *Moreover, if* a *does not divide* r *then* $q_t = q_h$ *and* $q^2 \equiv \pm 1 \pmod a$, *and so the associated circulant is arc-transitive.*

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Theorem

Let Γ, G, $\mathcal{B}$ *be as above. Then* $K_G(\Gamma_{\mathcal{B}}) = K_G(\mathrm{Alt}_G(\Gamma))$.

Theorem

Let Γ be a tetravalent G-HAT graph for some $G \le \mathrm{Aut}(\Gamma)$ and let $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. Let $K = K_G(\mathrm{Alt}_G(\Gamma))$ be the kernel of the action of G on the graph $\mathrm{Alt}_G(\Gamma)$ of G-alternating cycles of Γ. Then one of the following holds:

(i) $a = 2r$ *and* $K = D_r$;

(ii) $a = r = 2$, *in which case Γ is the wreath graph $C_n[2K_1]$ for some integer n, and K is isomorphic to a subgroup of the elementary abelian 2-group of order 2^n;*

(iii) $a = r > 2$ *and K is isomorphic to the dihedral group of order $2a$;*

(iv) $a < r$ *with $a \mid r$ and K is isomorphic to the cyclic group of order a, unless possibly if $a = 2$, in which case the kernel K can be trivial;*

(v) $a < r$ *with $a \nmid r$ and K is isomorphic to the cyclic group of order $a/2$.*

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Let Γ be a tetravalent G-HAT graph for some $G \leq \mathrm{Aut}(\Gamma)$ and let $r = \mathrm{rad}_G(\Gamma)$, $a = \mathrm{att}_G(\Gamma)$ and $q = \mathrm{jum}_G(\Gamma)$. Suppose a does not divide r and $4 < a < r$. If $q = 1$ or the graph $\mathrm{Alt}_G(\Gamma)$ is bipartite, then the graph Γ is arc-transitive. In particular, if $4 < a < r$, a does not divide r and $q \neq a/2 - 1$, then Γ is arc-transitive.