8th PhD Summer School in Discrete Mathematics
Questions on Colva's lectures on July 2nd.

1. Prove that every $\sigma \in \mathrm{S}_n$ can be written as a product of pairwise disjoint cycles, and this decomposition is unique up to the order of the cycles.

2. Let $i_1, i_2, \dots, i_k$ be distinct elements of $\underline{n}$ and let $\pi \in \mathrm{S}_n$. Show that

$$\pi^{-1}(i_1\ i_2\ \dots i_k)\pi = (i_1^\pi\ i_2^\pi\ \dots\ i_k^\pi).$$

Deduce that elements of S_n are conjugate if and only if they have the same cycle structure.

3. Why do we not normally get an action of a group G on itself by defining $a^g := ga$? Show that $a^g = g^{-1}a$ gives a faithful regular action of G on itself.

4. Let a finite group G act on the set of all of its subgroups by conjugation. Verify that this is an action. What is the stabiliser of a subgroup H of G? Deduce that the number of conjugates of H in G divides $|G|$.

5. Let G be a group acting transitively on a set Ω, let H be a subgroup of G, and let G_α be a point stabiliser. Show that the following are equivalent:

$$(i)\ G = G_\alpha H; \quad (ii)\ G = HG_\alpha; \quad (iii)\ H \text{ is transitive.}$$

In particular, the only transitive subgroup of G that contains G_α is G itself.

6. Let F be the Fano plane, as drawn in lectures. Use the Orbit-Stabiliser Theorem to show that the automorphism group of F has order 168.

7. Two permutation groups $G \leq \mathrm{Sym}(\Omega)$ and $H \leq \mathrm{Sym}(\Gamma)$ are called *permutation isomorphic* if there exists a bijection $\lambda : \Omega \to \Gamma$ and a group isomorphism $\psi : G \to H$ such that for all $\alpha \in \Omega$ and $x \in G$

$$(\alpha^x)\lambda = (\alpha\lambda)^{x\psi}.$$

Let $\Omega = \Gamma$. Show that G and H are permutation isomorphic if and only if they are conjugate subgroups of $\mathrm{Sym}(\Omega)$.

8. Let G have two permutation representations ρ and τ to $\mathrm{Sym}(\Omega)$. Show that ρ and τ are equivalent if and only if for some $a \in \mathrm{Sym}(\Omega)$ and for all $g \in G$

$$g\rho = a^{-1}(g\tau)a.$$

Show that the images of equivalent representations are permutation isomorphic.

9. Show that A_5 has a *transitive* action on 6 points. Hence or otherwise, show that S_6 has two inequivalent transitive representations on 6 points, but that the images of the representations are permutation isomorphic.