Decomposing multigraphs into stars of varying sizes

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- decompositions are edge decompositions;
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When does a complete multigraph λK_n admit a decomposition into stars of sizes $[m_1, \ldots, m_t]$?

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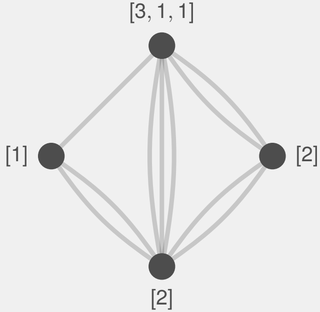

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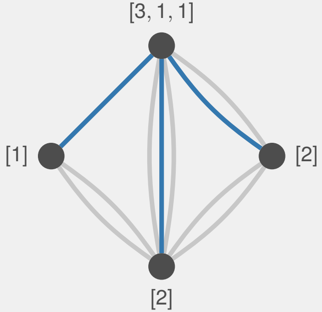

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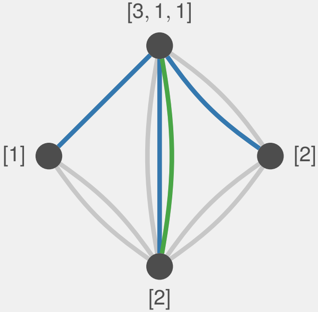

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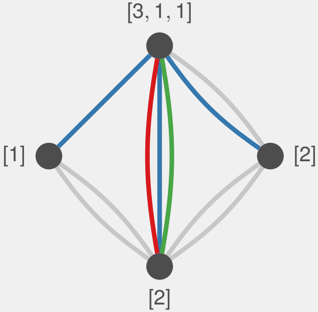

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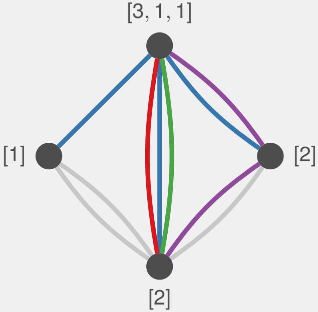

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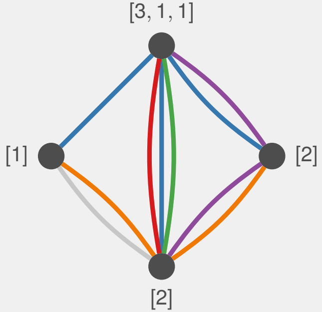

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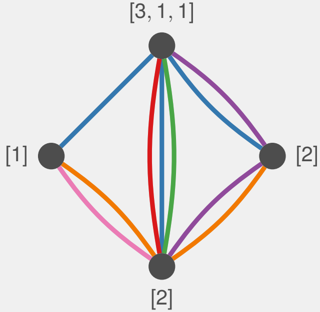

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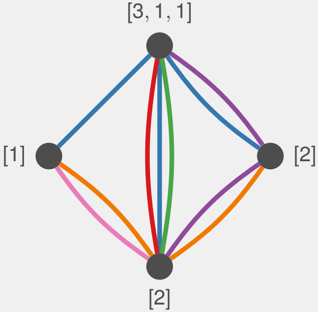

Hoffman (1994) answered this question in the case $m_1 = \cdots = m_t$.

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A decomposition of a multigraph G into stars of sizes $[m_1, \ldots, m_t]$ where each star has a specified centre exists if and only if $m_1 + \cdots + m_t = |E(G)|$ and no multiset of sizes is *overfull*.

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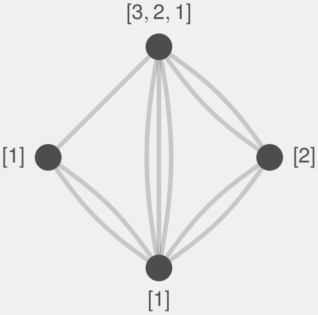

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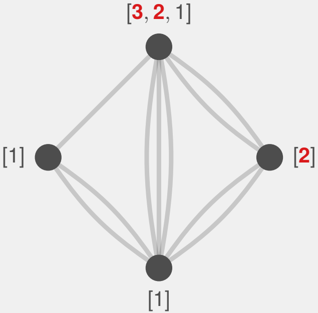

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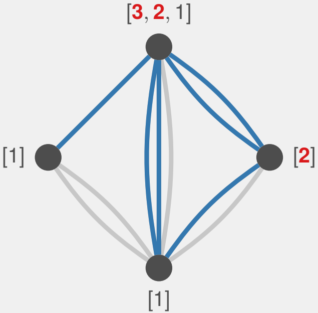

- Consider the red star sizes.
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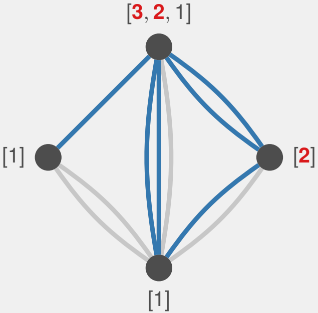

- Consider the red star sizes.
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- The red sizes sum to 7.

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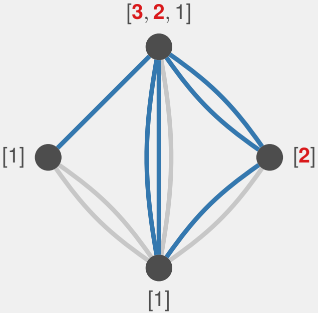

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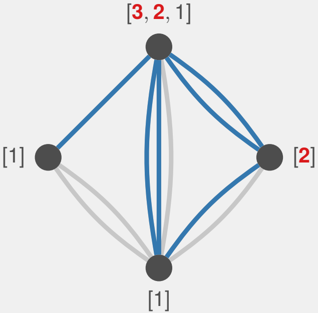

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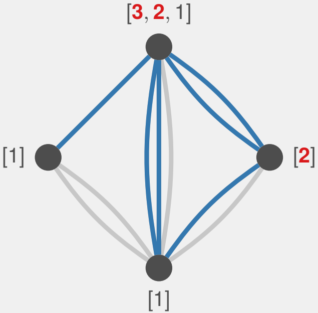

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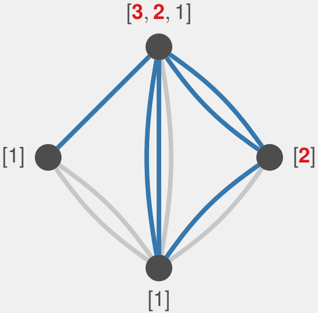

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Both results are proved using max-flow min-cut arguments.

Question 2: Star decompositions of complete multigraphs

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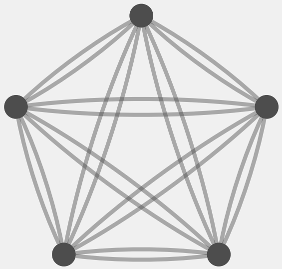

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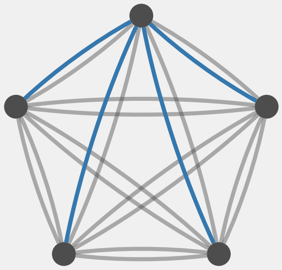

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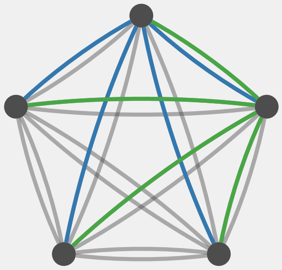

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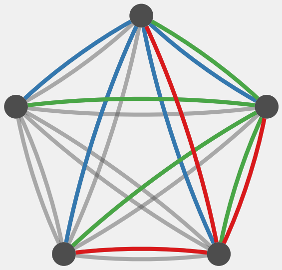

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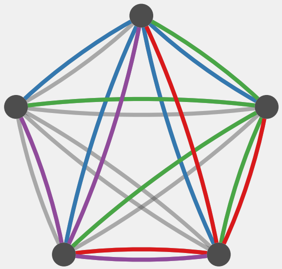

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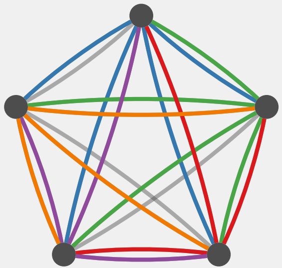

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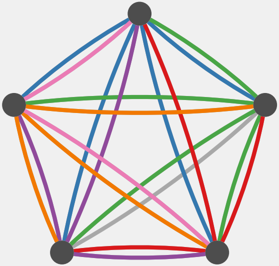

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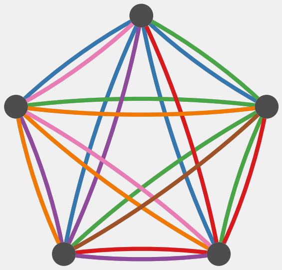

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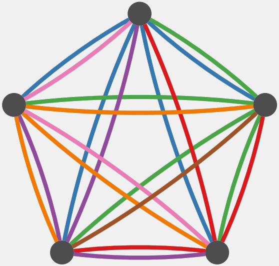

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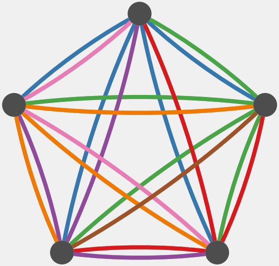

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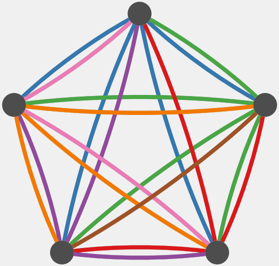

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Both results give simple numerical necessary and sufficient conditions for the existence of a decomposition.

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For any $\lambda \geqslant 2$, the problem of being given n and $[m_1, \ldots, m_t]$ and determining whether λK_n has a decomposition into stars of sizes $[m_1, \ldots, m_t]$ is NP-complete.

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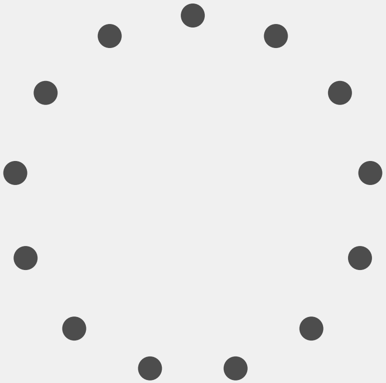

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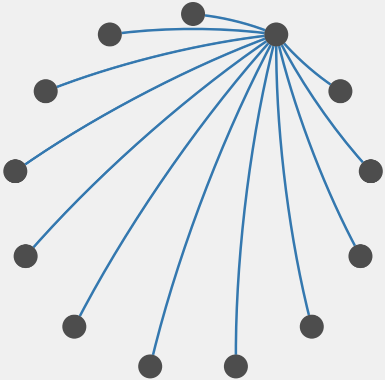

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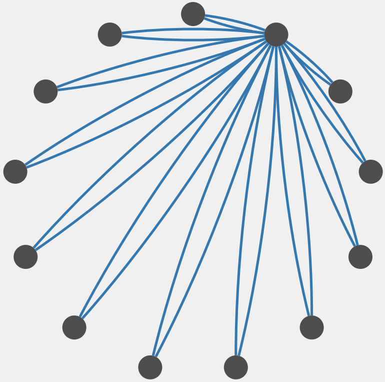

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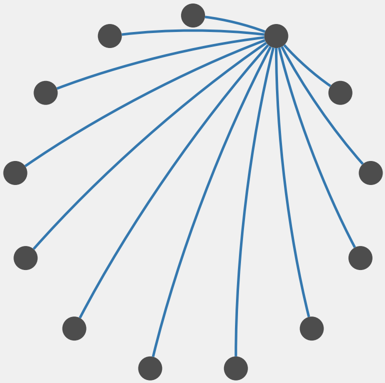

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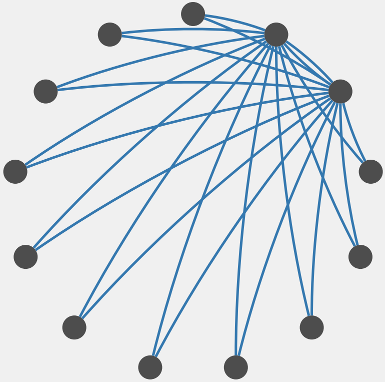

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For any $\lambda \geqslant 2$, the problem of being given n and $[m_1, \ldots, m_t]$ and determining whether λK_n has a decomposition into stars of sizes $[m_1, \ldots, m_t]$ is NP-complete.

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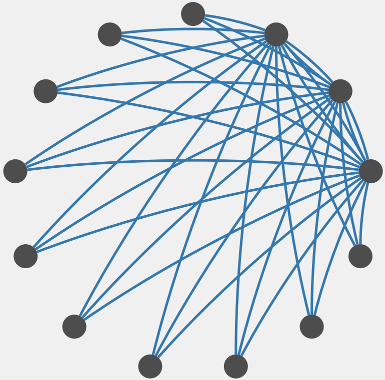

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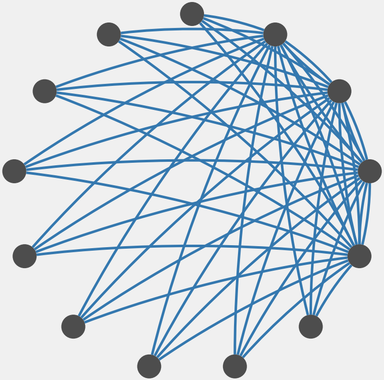

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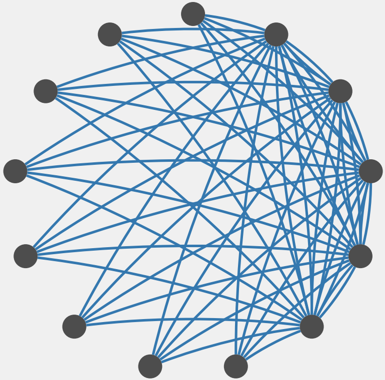

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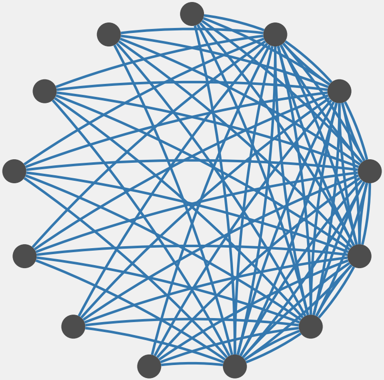

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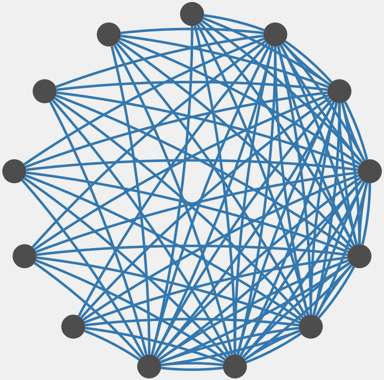

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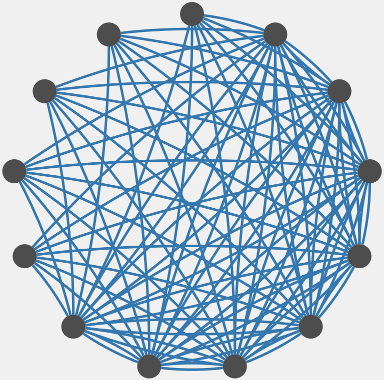

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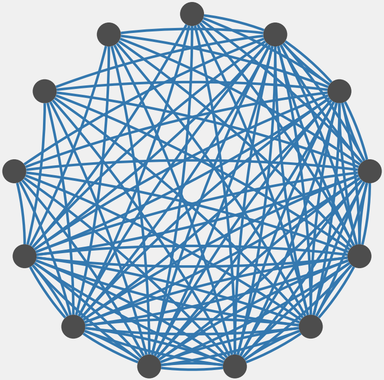

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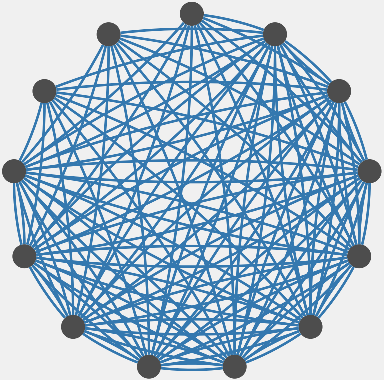

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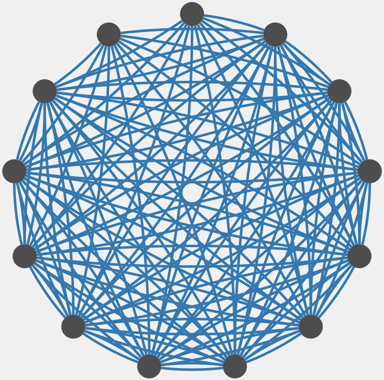

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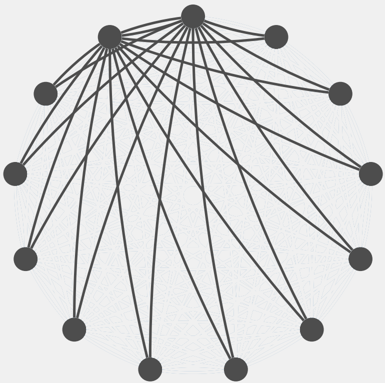

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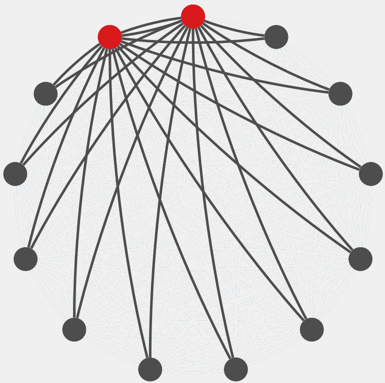

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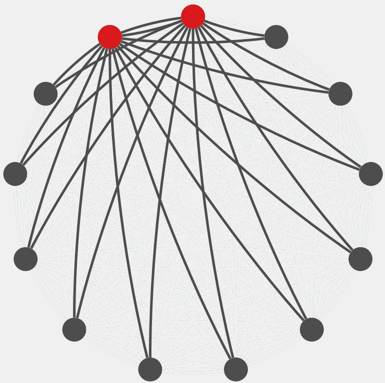

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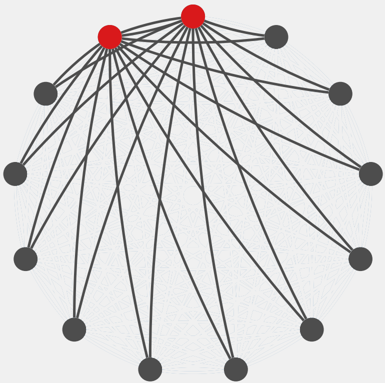

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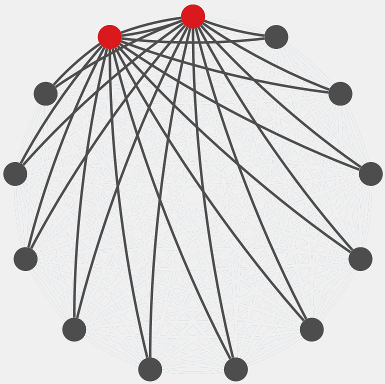

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This allows us to reduce PARTITION to our problem.

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What about if we limit the maximum star size?

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(λ, α)-STAR DECOMP

Instance: Positive integers n and $[m_1, \ldots, m_t]$ such that $\max(m_1, \ldots, m_t) \leqslant \alpha(n-1)$ and $m_1 + \cdots + m_t = \lambda\binom{n}{2}$.

Question: Does λK_n have a decomposition into stars of sizes $[m_1, \ldots, m_t]$?

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Theorem wannabe Cameron, Horsley

Let $\lambda \geqslant 2$ be an integer. Then (λ, α)-STAR DECOMP is NP-complete if and only if $\alpha > \alpha'$, where

$$\alpha' = \begin{cases} \frac{\lambda}{\lambda+1}, & \text{if } \lambda \text{ is odd;} \\ 1 - 4(\sqrt{\lambda(\lambda+2)} + 2)^{-2}, & \text{if } \lambda \text{ is even.} \end{cases}$$

Furthermore, if $\alpha \leqslant \alpha'$ then, for all sufficiently large n, the answer to (λ, α)-STAR DECOMP is affirmative.

Threshold configuration for λ odd

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Take a list $[m^{(\lambda+1)n/2-2}, \text{small stuff}]$, where $m = (\alpha' + \epsilon)(n-1)$.

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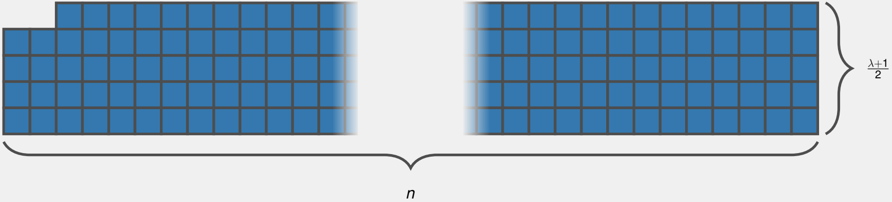
We can show the m s must be arranged as follows (otherwise the set of m s will be overfull):



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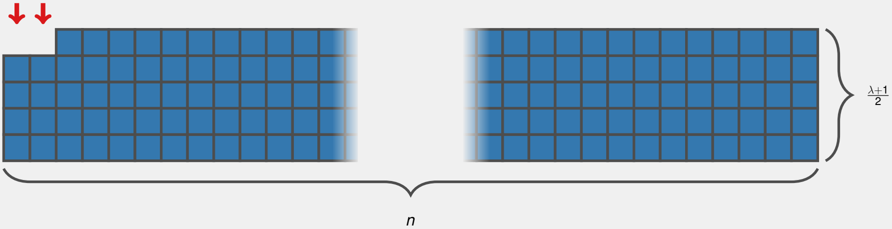

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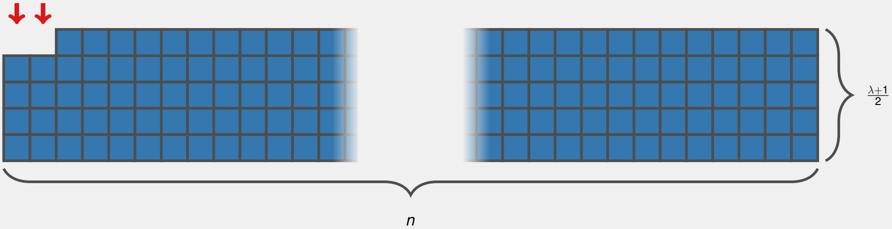

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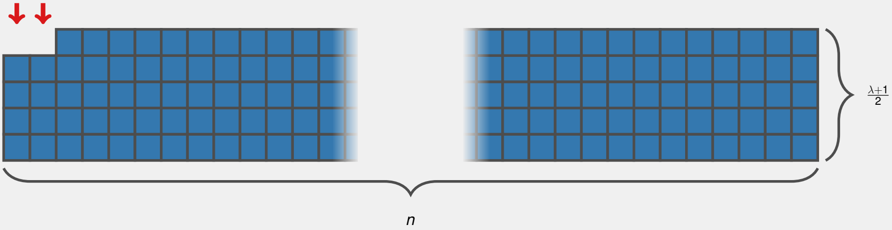

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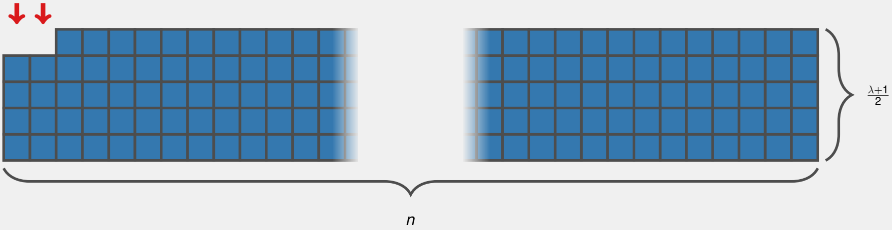

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- c obeys $\frac{\lambda}{\lambda+2}m < c < m$
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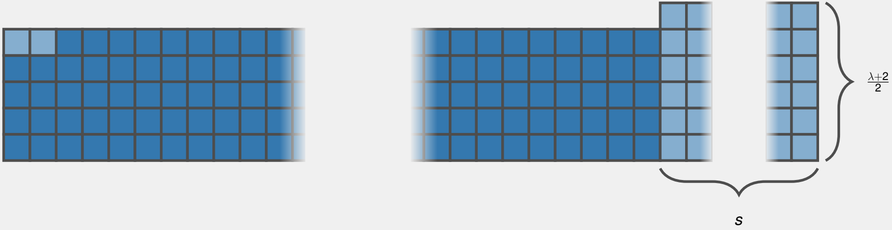

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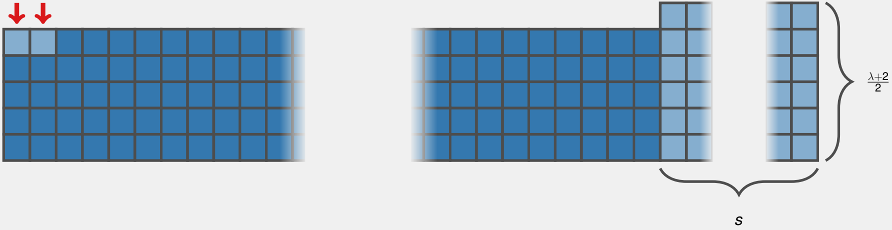

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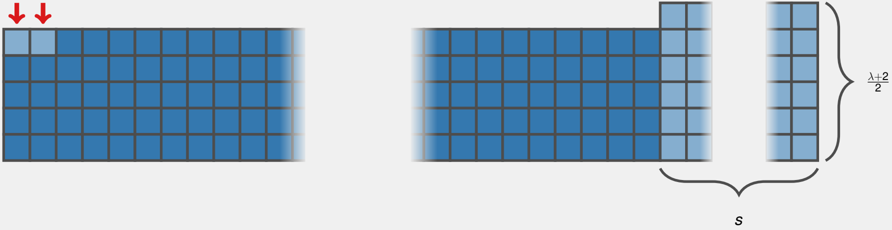

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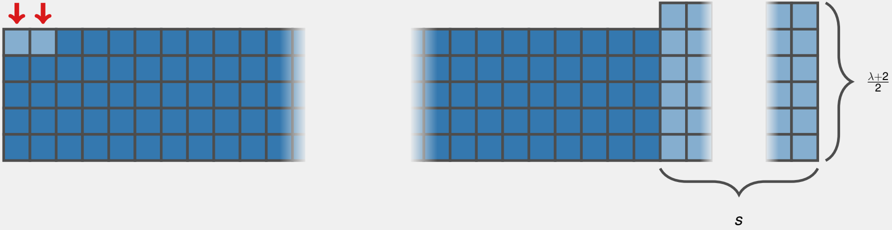

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