Orientable quadrilateral embeddings of cartesian products

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Quadrilateral embeddings and cartesian products

Orientable surfaces:

S_h

Embedding $\Phi : G \to \Sigma$: draw G in Σ without edge crossings.

Quadrilateral: open disk face bounded by 4-cycle.

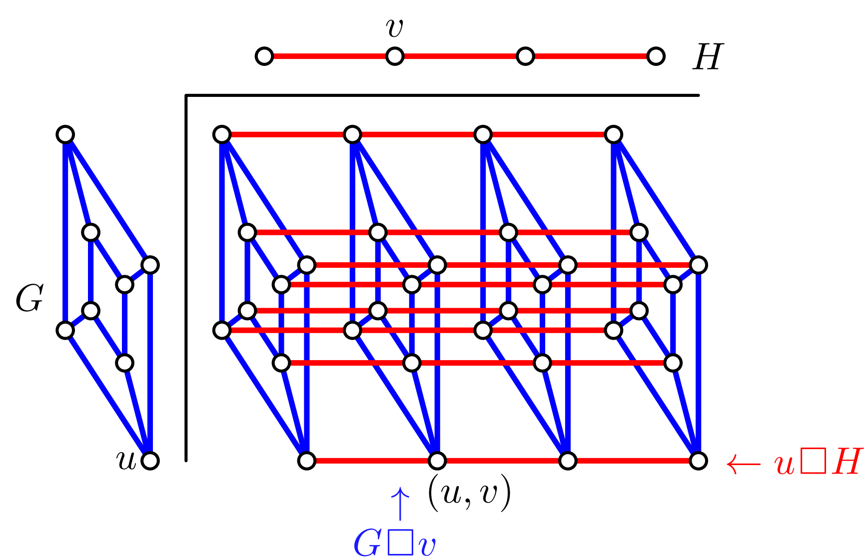
Quadrilateral embedding (QE): every face quadrilateral. So cellular.

Why quadrilateral embeddings? Minimum genus if graph has girth 4 or more.

Cartesian product (CP) $G \square H$:

G-edges inside $G \square v$, H-edges inside $u \square H$.

Why cartesian products? Many 4-cycles, improves chances of finding quadrilateral embedding.



Pisanski's three questions, 1992

Question 1: If G, H are arbitrary 1-factorable t-regular graphs, does $G\Box H$ always have an orientable quadrilateral embedding?
True if G, H bipartite (Pisanski, 1980).

Question 2: For t-regular G, $t \geq 2$, does
$$G\Box C_{2n_1}\Box C_{2n_2}\Box \ldots \Box C_{2n_t}$$
have an orientable quadrilateral embedding?
More general than $G\Box Q_{2t} = G\Box(\Box^t C_4)$.
True if G bipartite (Pisanski, 1980).

Question 3: For an arbitrary graph G, does $G\Box Q_n$ have an orientable quadrilateral embedding for all sufficiently large n? (Q_n = $\Box^n K_2$, hypercube.)
True if G bipartite, for $n \geq \Delta(G)$ (Pisanski, 1980).
True for regular G if $n \geq 2\Delta(G) + 3$ (Pisanksi, 1992).
Also true for all G if we drop 'orientable' (Pisanski, 1982 and also Hunter and Kainen, 2007).

Here we discuss Questions 2 and 3 ...

Our construction

Generalizes Pisanski's +/- construction, 1992.

Pisanski showed that for every t-regular G, there is an orientable QE of $G \square Q_n$ for all $n \geq 2t+3$.

- Begin with orientable emb. Φ of any graph G.

Add semiedges coloured by D, $|D| = r$: Φ^+ where

(0) each colour appears once at each vertex,

(1) edge/semiedge adjacency condition

($\to$ GH-faces),

(2) faces without semiedges are quadrilaterals

($\to$ G-faces).

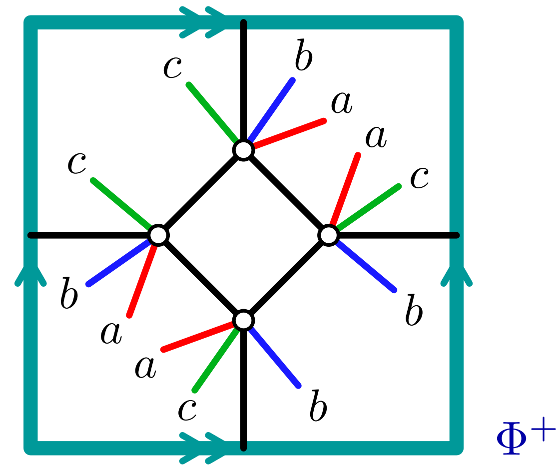
- Colour edges of r-regular bipartite H with D so

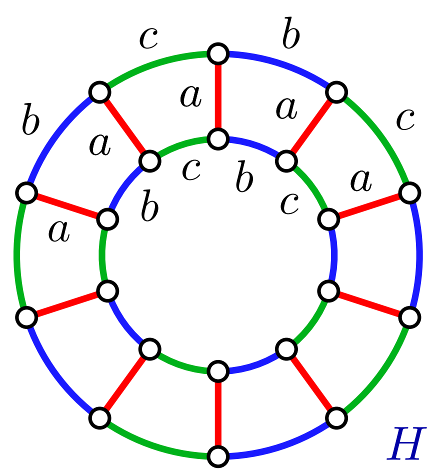
(3) consecutive colours d_1, d_2 in Φ^+ mean

$H(d_1, d_2)$ is a 4-cycle 2-factor ($\to$ H-faces).

- Use to derive orientable QE of $G \square H$.

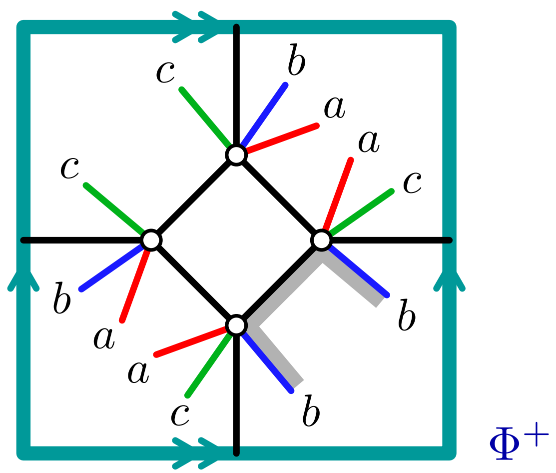
Example: $K_4 \square (C_{10} \square K_2)$

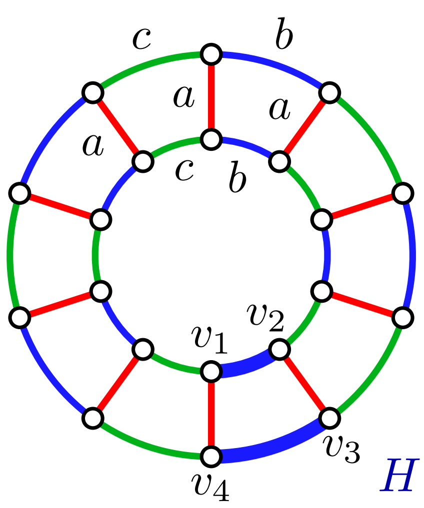




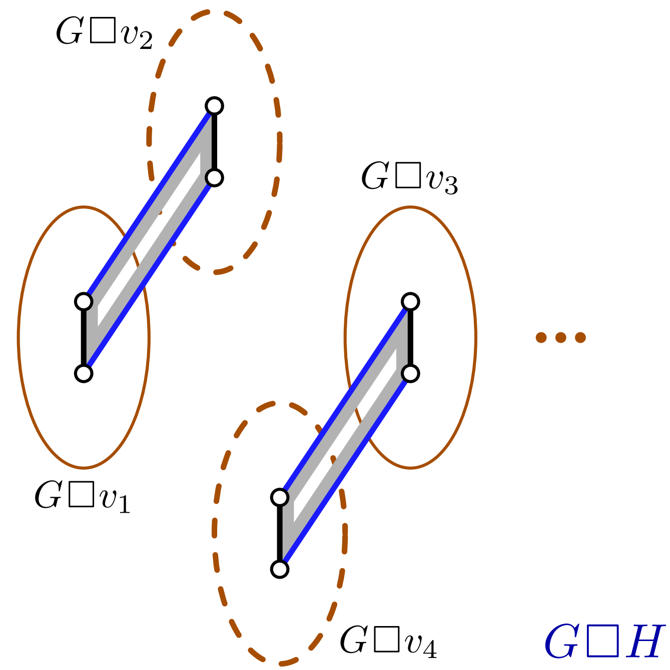
Construction details I

Example: $K_4 \square (C_{10} \square K_2)$



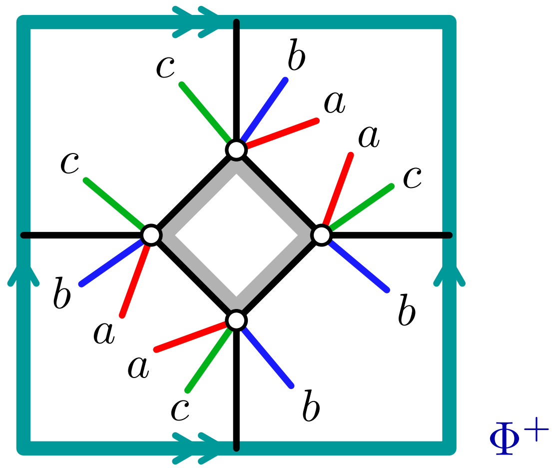


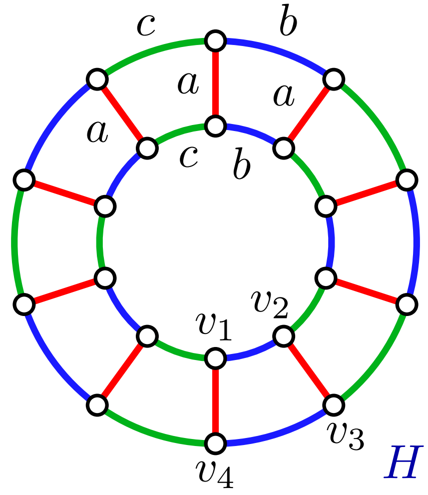
(1) Get GH-faces from corners between edges and semiedges, using edge/semiedge adjacency condition.



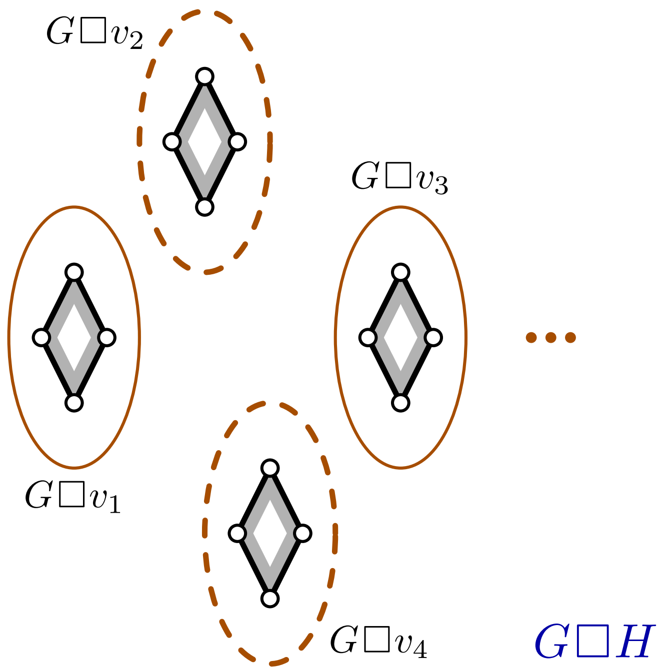
Construction details II

Example: $K_4 \square (C_{10} \square K_2)$



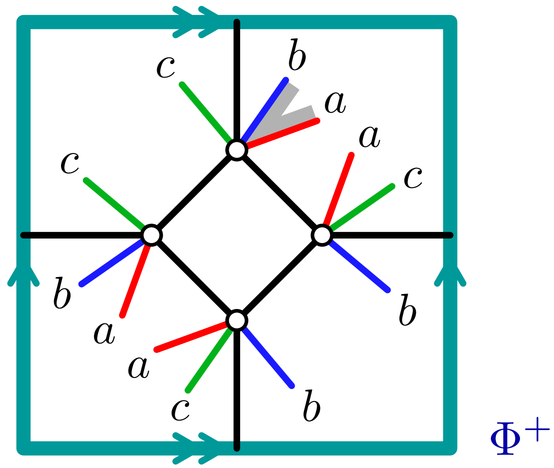


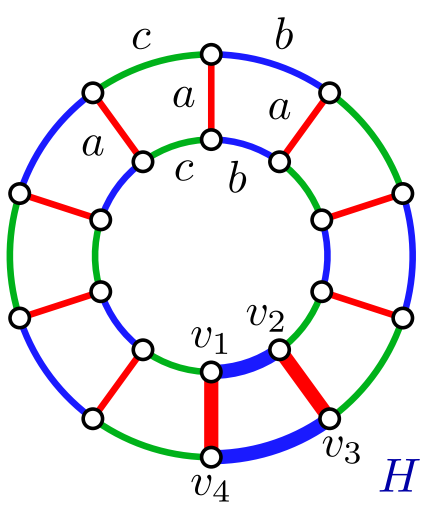
(2) Get G-faces from corners between pairs of edges, using fact that faces without semiedges are quadrilaterals.



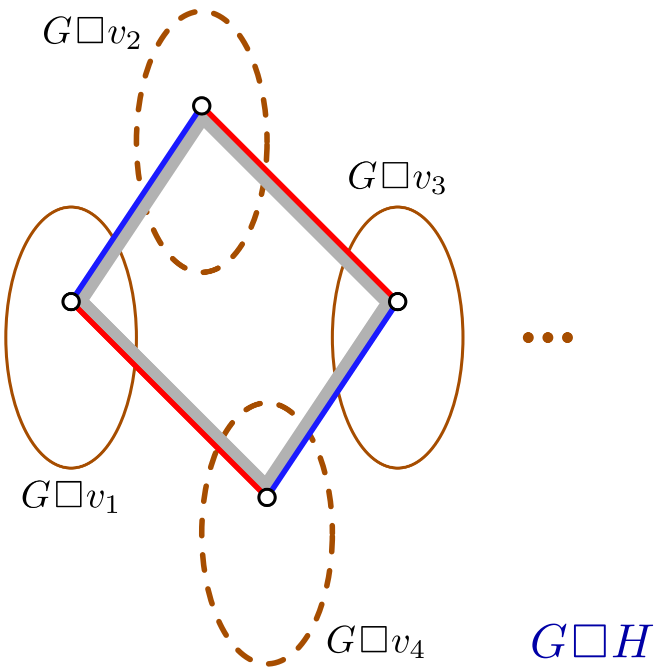
Construction details III

Example: $K_4 \square (C_{10} \square K_2)$





(3) Get H-faces from corners between pairs of semiedges, using fact that consecutive colours d_1, d_2 in Φ^+ mean $H(d_1, d_2)$ is a 4-cycle 2-factor.

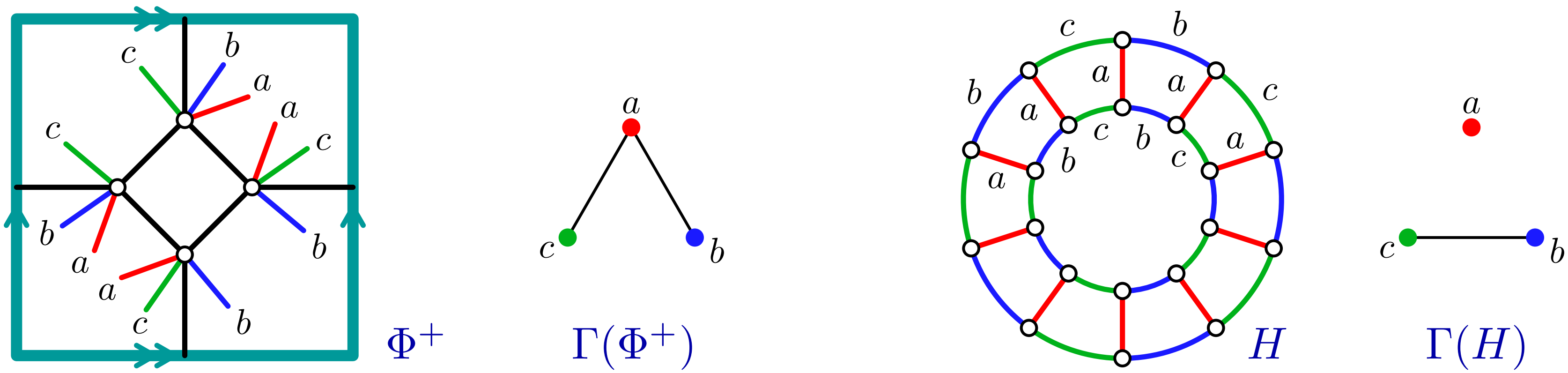


Conflict graphs

Hardest part is satisfying (3). Think of Φ^+ and H as generating conflicts between pairs of colours:

- ◦ conflict in Φ^+ if d_1, d_2 consecutive somewhere,
- ◦ conflict in H if $H(d_1, d_2)$ not a 4-cycle 2-factor.

Want conflict graphs $\Gamma(\Phi^+)$, $\Gamma(H)$ to be edge-disjoint. For example:



- Can weaken this. Enough if $\Gamma(\Phi^+)$ and $\Gamma(H)$ pack: one isomorphic to subgraph of complement of other. Can also use different colours for Φ^+, H.
- If H is itself a cartesian product of regular graphs all of the same degree (e.g., H is a cube) then we can use equitable colourings of $\Gamma(\Phi^+)$ to show that $\Gamma(\Phi^+)$ and $\Gamma(H)$ pack: Hajnal-Szemerédi Theorem or special construction.

Solving Questions 2 and 3

From equitable colourings we get:

Theorem: Suppose that G is k-edge-colourable, $k \geq 3$, and $H_1, H_2, \ldots, H_m$ are all s-regular bipartite graphs, where $m \geq 3$ and $sm \geq \lceil 3k/2 \rceil$. Then $G \square (H_1 \square H_2 \square \ldots \square H_m)$ has an orientable quadrilateral embedding.

Question 2: For t-regular G, $t \geq 2$, does

$$G \square C_{2n_1} \square C_{2n_2} \square \ldots \square C_{2n_t}$$

have an orientable quadrilateral embedding?

Answer: Yes, for $t \geq 3$. In fact, works for

$$G \square C_{2n_1} \square C_{2n_2} \square \ldots \square C_{2n_m}$$

provided $t \geq 2$ and $m \geq \max(3, \lceil 3(t+1)/4 \rceil)$.

Question 3: For an arbitrary graph G, does $G \square Q_n$ have an orientable quadrilateral embedding for all sufficiently large n?

Answer: Yes. Just take all $H_i = K_2$, then $n \geq \max(3, \lceil 3\chi'(G)/2 \rceil)$ works. ($\chi'(G)$, chromatic index, is $\Delta(G)$ or $\Delta(G) + 1$.)

Future directions

- Extend our construction for $G \square H$:
  - Nonorientable embeddings: start with nonorientable embedding of G, or use nonbipartite H.
  - Nonregular graphs H, using partial 4-cycle patchworks, or directly.
- What about orientable quadrilateral embeddings of $G \square H$ when neither G nor H is bipartite? Nothing much known.
- We have 3-regular counterexamples to Question 1: no orientable QE of $G \square H$ for G, H both t-regular, 1-factorable. Find counterexamples for Question 1 that are t-regular for $t \geq 4$. Should be doable.
- What about Question 2 for 2-regular G? Does $C_{\mathrm{odd}} \square C_{\mathrm{even}} \square C_{\mathrm{even}}$ have an orientable quadrilateral embedding? Our technique does not work.

Thank you!

And congratulations to Brian and Dragan!