Local actions in arc-transitive graphs

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Groups, Graphs and More
(Koper, Slovenia, 27-31 May 2018)

Questions asked by Pierre-Emmanuel Caprace

When? August 2017

Where? Groups St Andrews conference, Birmingham (UK)

What? Does there exist a 2-transitive permutation group P such that only finitely many simple groups act arc-transitively on a connected graph X with local action P (i.e. with vertex-stabiliser inducing P on the neighbourhood of the vertex)?

Follow-up question (special case): Can P be A_6?

Further context: These questions were conveyed to others by Gabriel Verret and Michael Giudici at a 'Tutte Centenary Retreat' workshop in November 2017.

Background on arc-transitive graphs

A non-trivial graph X is arc-transitive (AT) (or symmetric) if its automorphism group Aut(X) is transitive on ordered pairs of adjacent vertices, or in other words, if X is vertex-transitive and the stabiliser in Aut(X) of any vertex of X is transitive on the neighbourhood $X(v)$ of v.

Examples

- Cycle graphs C_n, complete graphs K_n
- Complete bipartite graphs $K_{n,n}$ of constant valency
- Hypercube graphs Q_n
- The Petersen graph, Heawood graph, Tutte's 8-cage, etc.

A construction for arc-transitive graphs

Suppose X is a k-regular connected graph, and G is a subgroup of Aut(X) acting transitively on the arcs of X, and H is the stabiliser in G of a vertex v of X, and a is an automorphism that swaps v with one of its neighbours. Then

(1) $a^2 \in H$

(2) $K = H \cap a^{-1}Ha$ has index k in H, and

(3) $G = \langle H, a \rangle$.

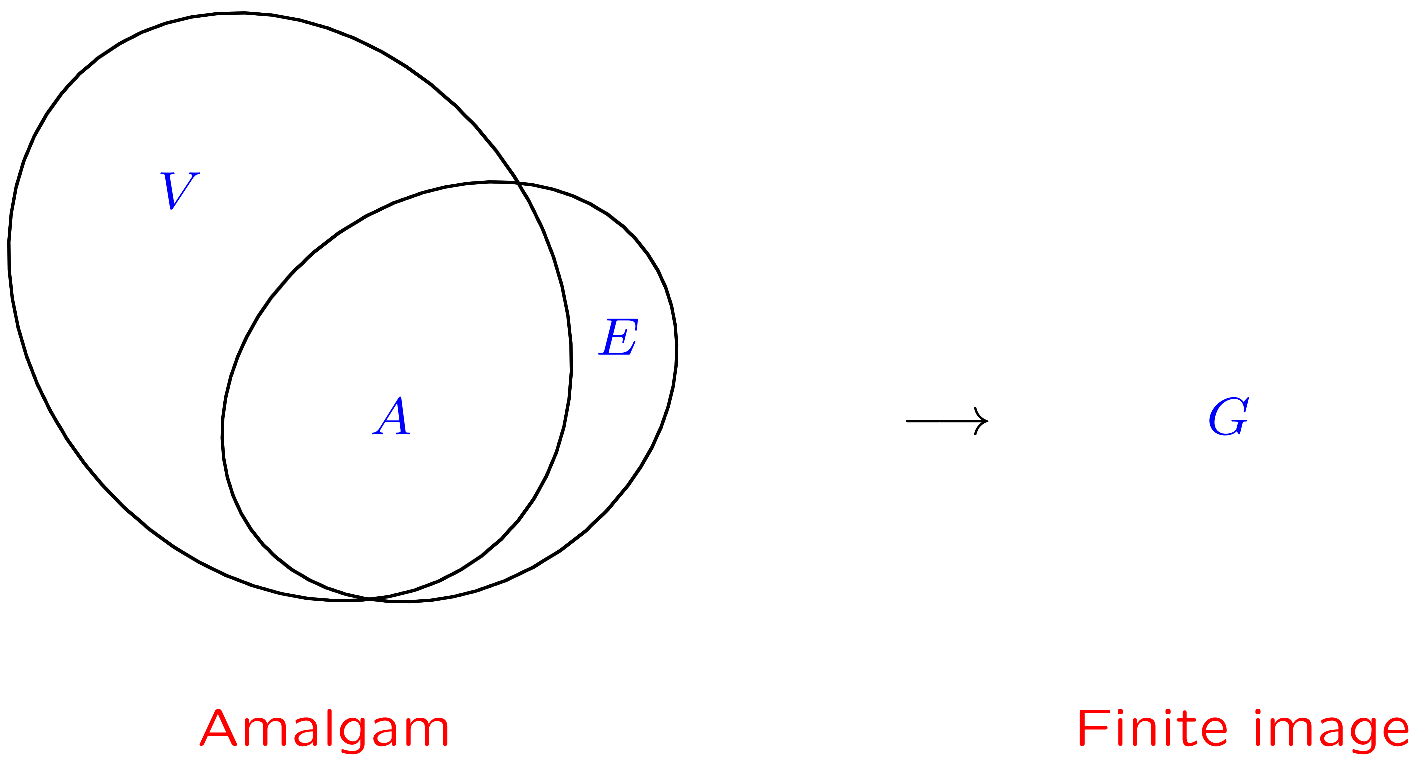
Conversely, if G is a finite group, H is a subgroup of G, and a is an element of G satisfying conditions (1) to (3) above, define a graph $X = X(G, H, a)$ with vertices Hx for $x \in G$, and edges $Hx - Hy$ whenever $xy^{-1} \in HaH$. Then X is k-regular and connected, and G acts arc-transitively on X with vertex-stabiliser H and arc-stabiliser $K = H \cap a^{-1}Ha$.

Connection with group amalgams

In the context given on the previous slide, if we let V, E and A be abstract groups isomorphic to the stabiliser H in $G \leq \mathrm{Aut}(X)$ of a vertex v of X, the stabiliser $K = H \cap a^{-1}Ha$ of the arc (v, w), and the stabiliser $\langle K, a \rangle$ of the edge $\{v, w\}$, then $G = \langle H, a \rangle = \langle H, K, a \rangle$ is a quotient of the free product $V *_A E$ of V and E with $A = V \cap E$ amalgamated.

Conversely, if G is the image of $V *_A E$ under any group homomorphism that is faithful on each of V and E (and A), then G acts arc-transitively on a graph X with vertex-, edge- and arc-stabiliser isomorphic to V, E and A respectively.

Note: $|V : A| = k$ (valency) and $|E : A| = 2$, and the local action of G on X is equivalent to the natural action of V on cosets of A. We'll assume $k \geq 3$ (otherwise G is dihedral).


V
E
A
G
Amalgam
Finite image

Conjecture [Džambić and Jones (2013), re-worded]

If V and E are any two finite groups with a common subgroup A with index $|V:A| \geq 3$ and index $|E:A| \geq 2$, then all but finitely many alternating groups A_n occur as homomorphic images of the amalgamated free product $V *_A E$.

Stronger conjecture [MC]

Let V and E be any two finite groups with a common subgroup A with index $|V:A| \geq 3$ and index $|E:A| \geq 2$, and let K be the core of A in $V *_A E$. Then all but finitely many A_n occur as the image of the amalgamated free product $V *_A E$ under some homomorphism that takes V and E to subgroups (of A_n) isomorphic to V/K and E/K respectively.

In particular, if $A = V \cap E$ is core-free in $V *_A E$, then the stronger conjecture suggests that all but finitely many A_n occur as images of $V *_A E$ under homomorphisms that are faithful on each of V and E.

[This is stronger since any quotient of the modular group $C_2 *_{C_1} C_3$ is also a quotient of $C_4 *_{C_2} D_3$, but not vice versa.]

Evidence in support of the stronger conjecture:

Lots of it! Alternating quotients of such amalgams have produced infinite families of finite arc-transitive, path-transitive and semi-symmetric 3-valent graphs, 7-arc-transitive 4-valent graphs, arc-transitive digraphs, chiral maps, chiral polytopes, and even hyperbolic 3-manifolds. [MC (1988–)]

Consequence

If the stronger form of the amalgam conjecture is true, then the answer to the main question by Caprace is "No".

In fact the 2-transitive hypothesis in Caprace's question could then be relaxed to transitive, giving the following, for example:

Likely Theorem

If P is any transitive permutation group of degree $k \geq 3$, then all but finitely many A_n act arc-transitively on a connected k-valent graph with local action P.

Some specific known cases

The stronger form of the amalgam conjecture is known to be true in many cases, including the following:

- $(V, A, E) = (C_k, 1, C_2)$ for all $k \geq 3$ [*c.f.* regular maps]
- $(V, A, E) = (D_k, C_2, V_4)$ for $k = 4$ and all $k \geq 7$
- $(V, A, E) = (A_5, C_5, D_5)$ [Džambić & Jones]

Hence the answer to Caprace's question is 'No' when P is a cyclic or dihedral group of degree ≥ 3, or the group of degree 12 induced by A_5 on cosets of a subgroup of order 5.

It is very clear that the same answer holds for many other permutation groups besides these.

Caprace's follow-up question: $P = A_6$

Here we take $(V, A) = (A_6, A_5)$ and $E = S_5$ or $A_5 \times C_2$ (the only groups containing A_5 as a subgroup of index 2). Let's take $E = A_5 \times C_2$ and consider the amalgam $A_6 *_{A_5} (A_5 \times C_2)$.

This has many alternating quotients, e.g. A_{13}, generated by

$x = (2, 3)(4, 7)(8, 11)(12, 13)$,
$y = (3, 4, 7, 10, 6)(5, 9, 12, 11, 8)$, and
$a = (1, 2)(3, 5)(4, 8)(6, 9)(7, 11)(10, 12)$.

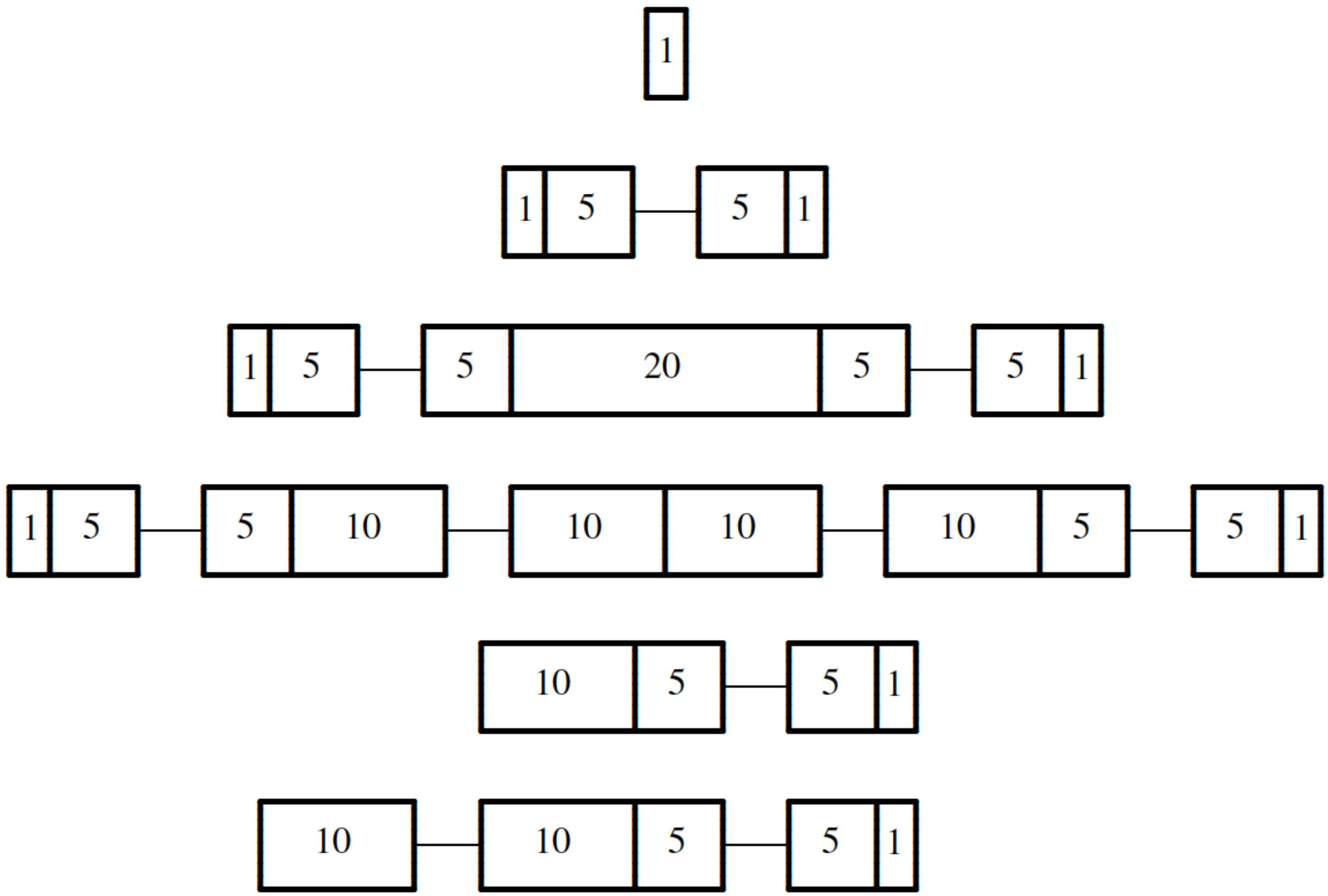
Here $\langle x, y\rangle$ is isomorphic to A_6, with orbits $\{1\}$, $\{2, 3, 4, 6, 7, 10\}$ and $\{5, 8, 9, 11, 12, 13\}$, and $\langle y, xy^{-1}xyx\rangle$ is isomorphic to A_5, with orbits $\{1\}$, $\{2\}$, $\{3, 4, 6, 7, 10\}$, $\{5, 8, 9, 11, 12\}$ and $\{13\}$, and conjugation of the latter subgroup by a behaves like an inner automorphism, centralising $xy^{-1}xyx$ and inverting y.

Note that the permutation a 'links' the orbits of $\langle y, xy^{-1}xyx\rangle$.

Construction for quotients of $A_6 *_{A_5} (A_5 \times C_2)$

We take transitive permutation representations of $V = A_6$, and use an involution a in $E \setminus A = (A_5 \times C_2) \setminus A_5$ to link them together by their 'compatible' orbits of $A = A_5$.

To answer Caprace's follow-up question, we can take as 'building blocks' just six different transitive permutation representations of $A_6 *_{A_5} (A_5 \times C_2)$, on 1, 12, 42, 62, 21 and 31 points, and combine these in different ways to obtain all but finitely many A_n and all but finitely many S_n as quotients.



1
1 5
5 1
1 5
5 20 5
5 1
1 5
5 10
10 10
10 5
5 1
10 5
5 1
10
10 5
5 1

For each of these six representations of $A_6 *_{A_5} (A_5 \times C_2)$, the image splits into orbits of the subgroups $V = \langle x, y \rangle \cong A_6$, $E = \langle y, u, a \rangle \cong A_5 \times C_2$ and $A = V \cap E = \langle y, u \rangle \cong A_5$.

For example, the degree 42 representation splits into three orbits of V, of lengths 6, 30 and 6, and these in turn split into seven orbits of A, of lengths 1, 5, 5, 20, 5, 5 and 1,

Every orbit of the subgroup $E = \langle A, a \rangle$ is then either an orbit of A preserved by a, or a union of two orbits of A that are interchanged by a.

Length 1 orbits of A preserved by a from different representations can be linked together by a to produce larger representations.

Construction (cont.)

Every *odd* positive integer $d \geq 395$ is expressible in the form $d = 21 + 12k + 62m$ for positive integers k and m.

We may now construct a transitive permutation representation of $A_6 *_{A_5} (A_5 \times C_2)$ of degree d by stringing together a single copy of the 21-point block with k copies of the 12-point block and m copies of the 62-point block.

In the resulting representation, the permutation induced by a is even when m is odd, but odd when m is even. Also another element w induces an even/odd permutation with cycle structure $1^3\, 2^{1+3k+8m}\, 4^3\, 5^1\, 6^{k-1+6m}\, 7^1\, 8^1\, 10^{m-1}$, and then w^{120} is a single 7-cycle, which can be used to prove that the image is A_d when m is odd, and S_d when m is even.

Next, we can add a copy of the 1-point block to the final copy of the 62-point block, and the same argument works, except that the parity of the permutations a and w changes, with a 5-cycle of w becoming another 6-cycle. Here the image is S_{d+1} when m is odd, and A_{d+1} when m is even.

Also we can replace the single copy of the 21-point block by a copy of the 31-point block and insert a single copy of the 42-point block, and get images S_{d+52} and A_{d+53} when m is odd, and A_{d+52} and S_{d+53} when m is even.

Finally, because $d = 21 + 12k + 62m \equiv 1 + 2m \bmod 4$, this means we have both A_n and S_n as images of $A_6 *_{A_5} (A_5 \times C_2)$, for all sufficiently large n, as required.

Corollary 1 All but finitely many A_n act arc-transitively on some 6-valent symmetric graph with vertex-stabiliser A_6.

Corollary 2 All but finitely many A_n occur as quotients of the amalgamated free product $A_6 *_{A_5} A_6$.

[This strengthens an observation made at the Groups St Andrews conference by Peter Neumann and Cheryl Praeger that $A_6 *_{A_5} A_6$ has infinitely many alternating quotients.]

Proof. In the group $V *_A E = A_6 *_{A_5} (A_5 \times C_2)$ used above, the subgroup $B = \langle V, V^a \rangle$ is isomorphic to $A_6 *_{A_5} A_6$ and has index 2, and hence also maps onto A_n for every large n.

THANK YOU

And finally ...

An advertisement for the 41st Australasian Conference on Combinatorial Mathematics and Combinatorial Computing the week 10-14 December 2018 in Rotorua, New Zealand