Patterns of Mirrors on Quasi-Platonic Surfaces

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Graphs, groups and more: celebrating Brian Alspach's 80th and Dragan Marušič's 65th birthdays
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Platonic Surfaces

It is known that every compact Riemann surface X of genus g can be expressed in the form $\mathbb{U}/\Omega$, where $\mathbb{U}$ is the Riemann sphere Σ, the Euclidean plane $\mathbb{C}$, or the hyperbolic plane $\mathbb{H}$, depending on whether g is 0, 1 or > 1, respectively, and Ω is a discrete group of isometries of $\mathbb{U}$.

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If Ω is normal in the ordinary triangle group $\Gamma[2, m, n]$, which has a presentation

$$\langle x, y, z \mid x^2 = y^m = z^n = xyz = 1 \rangle,$$

then X is called Platonic.

A map $\mathcal{M}$ on a Riemann surface X is an embedding of a finite connected graph $\mathcal{G}$ into X such that the components of $X \backslash \mathcal{G}$ are open discs, which are called the faces of $\mathcal{M}$.

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An automorphism of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence.

$\mathrm{Aut}^{\pm}(\mathcal{M})$: Group of all automorphisms of $\mathcal{M}$

$\mathrm{Aut}^{+}(\mathcal{M})$: Group of orientation-preserving automorphisms of $\mathcal{M}$

If $\mathrm{Aut}^{+}(\mathcal{M})$ is transitive on the directed edges, then $\mathcal{M}$ is called regular. If $\mathrm{Aut}^{\pm}(\mathcal{M})$ is transitive on the flags, then $\mathcal{M}$ is called reflexible.

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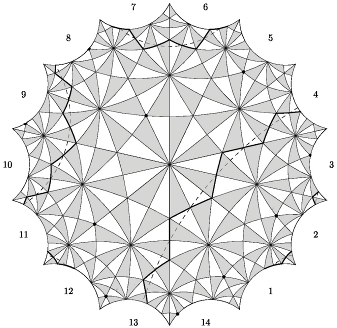
Example

Klein's surface of genus 3 underlies a regular map $\mathcal{M}$ of type $\{3, 7\}$. This surface contains 28 mirrors fixed by the reflections of $\mathcal{M}$ and these mirrors divide it into 336 (2,3,7)-triangles.

Klein's surface of genus 3



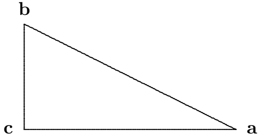
F. Klein, *Über die Transformation siebenter Ordnung der elliptischen Funktionen*, Math. Ann. **14** (1879), 428–471.

Kleins's Notation

b : vertex

c : edge center

a : face center



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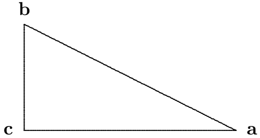
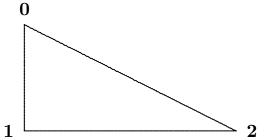


Coxeter's Notation

0 : vertex

1 : edge center

2 : face center



Every mirror M of a reflection of a regular map $\mathcal{M}$ passes through some of the geometric points of $\mathcal{M}$ such that these points form a periodic sequence, which is called the pattern of M. (By geometric points we mean the vertices, the face-centers and the edge-centers of $\mathcal{M}$.)

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Example

Every mirror on Klein's surface of genus 3 has pattern **010212010212010212** which we abbreviate to $(\mathbf{010212})^3$. Here **010212** is a link and the link index is 3.

[Melekoğlu-Singerman, 2016]:

- (i) The pattern of any mirror in a regular map $\mathcal{M}$ of type $\{m, n\}$ is obtained from one of the six links **01**, **02**, **12**, **0102**, **0212** and **010212**;
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The possible patterns according to the parity of m and n are given in the following table.

Table : Patterns

Case	Pattern
m and n odd	$(\mathbf{010212})^{\ell}$
m odd n even	$(\mathbf{01})^{\ell_1}$
m odd n even	$(\mathbf{0212})^{\ell_2}$
m even n odd	$(\mathbf{12})^{\ell_1}$
m even n odd	$(\mathbf{0102})^{\ell_2}$
m and n even	$(\mathbf{01})^{\ell_1}$
m and n even	$(\mathbf{12})^{\ell_2}$
m and n even	$(\mathbf{02})^{\ell_3}$

Here ℓ, ℓ_1, ℓ_2 and ℓ_3 are the link indices and ℓ_is in different lines need not be equal.

Quasi-Platonic Surfaces

Now let $X = \mathbb{U}/\Omega$ be a compact Riemann surface. If Ω is normal in the ordinary triangle group $\Gamma[l, m, n]$, which has a presentation

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Let Ω be also normal in the extended triangle group $\Gamma(l, m, n)$, which has a presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^l = (bc)^m = (ca)^n = 1 \rangle.$$

Then X can be divided into (l, m, n)-triangles.

As described in [Corn-Singerman, 1988], X can also be divided into l, m and $2n$ sided regular polygons, which are hypervertices, hyperedges and hyperfaces of a reflexible regular hypermap $\mathcal{H}$ of type (l, m, n) contained by X.

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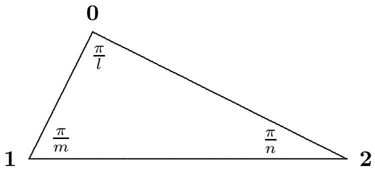
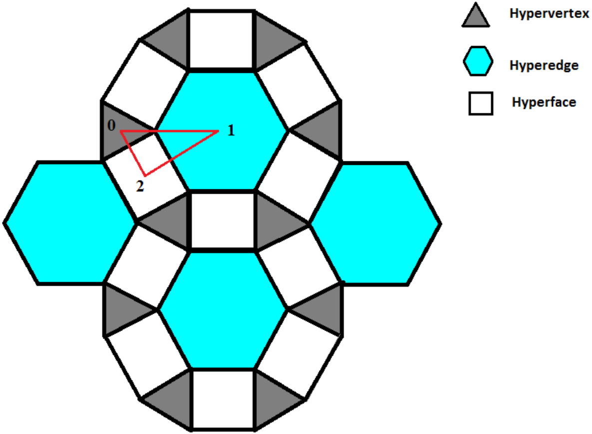
Again, every corner of a (l, m, n)-triangle is either a hypervertex, hyperedge or hyperface $\mathcal{H}$, and we use the same notation.

0 : Hypervertex center

1 : Hyperedge center

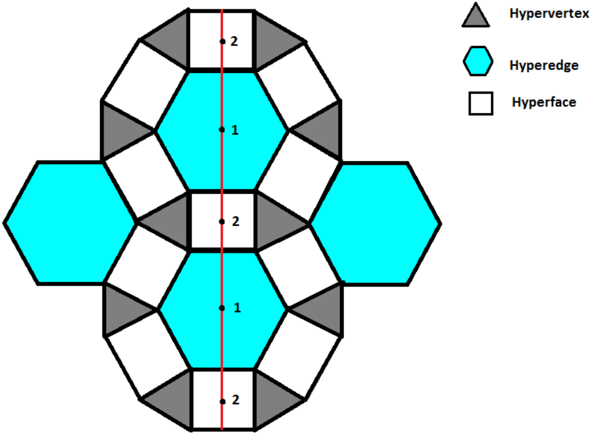
2 : Hyperface center



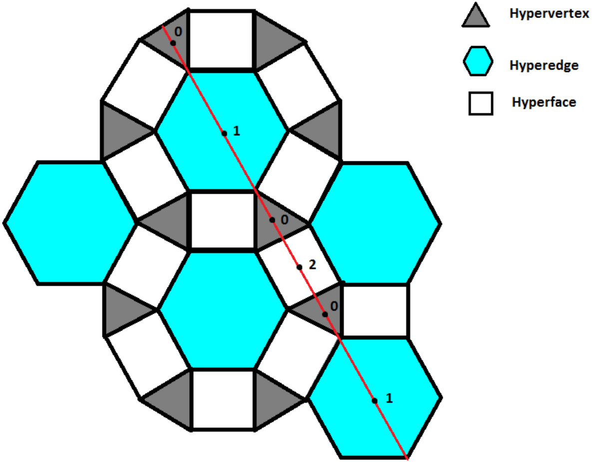


A hypermap of type (3,6,2) and a (3,6,2)-triangle

Patterns of Mirrors



A mirror with pattern $(\mathbf{12})^{\ell}$



A mirror with pattern $(\mathbf{0102})^{\ell}$

Table : Patterns of Hypermaps

Case	l	m	n	Pattern
1	even	even	even	$(\mathbf{02})^{\ell_1}$, $(\mathbf{01})^{\ell_2}$, $(\mathbf{12})^{\ell_3}$
2	odd	even	even	$(\mathbf{0102})^{\ell_1}$, $(\mathbf{12})^{\ell_2}$
3	even	odd	even	$(\mathbf{1012})^{\ell_1}$, $(\mathbf{02})^{\ell_2}$
4	even	even	odd	$(\mathbf{0212})^{\ell_1}$, $(\mathbf{01})^{\ell_2}$
5	even	odd	odd	$(\mathbf{012021})^{\ell}$
6	odd	even	odd	$(\mathbf{010212})^{\ell}$
7	odd	odd	even	$(\mathbf{020121})^{\ell}$
8	odd	odd	odd	$(\mathbf{012})^{\ell}$, $(\mathbf{210})^{\ell}$

Mirror Automorphisms

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Let M be a mirror of a reflection of a $\mathcal{H}$. A mirror automorphism of M is an automorphism of $\mathcal{H}$ that cyclically permutes the links of the pattern of M.

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[Melekoğlu-Singerman, 2016]:

Each pattern corresponds to a conjugacy class of mirror automorphisms, and the order of the mirror automorphisms in that conjugacy class is equal to the corresponding link index.

In the following table, we give a representative mirror automorphism for each pattern (link). In the table, A, B and C are the generators of $\mathrm{Aut}^+(\mathcal{H})$ given below:

$$\langle A, B, C \mid A^l = B^m = C^n = ABC = \cdots = 1 \rangle$$

Note that each mirror automorphism is written as a product of two orientation-preserving involutions. Also, when we have a regular map, only the first six rows occur.

Table : Mirror Automorphisms

Case	Link	Mirror Automorphism
M 1	**01**	$A^{\frac{l}{2}} B^{\frac{m}{2}}$
M 2	**02**	$A^{\frac{l}{2}} C^{\frac{n}{2}}$
M 3	**12**	$B^{\frac{m}{2}} C^{\frac{n}{2}}$
M 4	**0102**	$B^{\frac{m}{2}} A^{\frac{l-1}{2}} C^{\frac{n}{2}} A^{\frac{l+1}{2}}$
M 5	**0212**	$B^{\frac{m}{2}} C^{\frac{n+1}{2}} A^{\frac{l}{2}} C^{\frac{n-1}{2}}$
M 6	**010212**	$B^{\frac{m}{2}} C^{\frac{n+1}{2}} A^{\frac{l+1}{2}} B^{\frac{m}{2}} A^{\frac{l-1}{2}} C^{\frac{n-1}{2}}$
7	**1012**	$C^{\frac{n}{2}} B^{\frac{m-1}{2}} A^{\frac{l}{2}} B^{\frac{m+1}{2}}$
8	**012021**	$A^{\frac{l}{2}} B^{\frac{m+1}{2}} C^{\frac{n+1}{2}} A^{\frac{l}{2}} C^{\frac{n-1}{2}} B^{\frac{m-1}{2}}$
9	**020121**	$C^{\frac{n}{2}} B^{\frac{m-1}{2}} A^{\frac{l-1}{2}} C^{\frac{n}{2}} A^{\frac{l+1}{2}} B^{\frac{m+1}{2}}$
10	**012**	$A^{\frac{l+1}{2}} B^{\frac{m+1}{2}} C^{\frac{n+1}{2}}$
11	**210**	$C^{\frac{n-1}{2}} B^{\frac{m-1}{2}} A^{\frac{l-1}{2}}$

Thank You