On Reflexible Polynomials

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Joint work with
Boštjan Kuzman and Primož Potočnik

Graphs, groups, and more:
celebrating Brian Alspach's 80th and Dragan Maruščč's 65th birthdays
Koper, Slovenia

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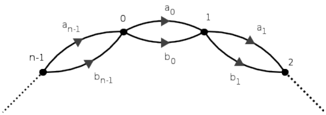
Classify Γ when $\Gamma/\mathbb{Z}_p^r = K_1, K_2, C_n$

Completely solved, except for $\Gamma/\mathbb{Z}_p^r = C_n$ and p odd

Minimal $\mathbb{Z}_p^r$-coverings $\Gamma \to C_n^{(2)}$



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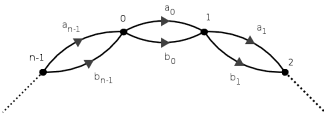

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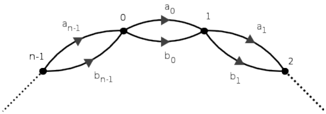

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M, Marušič, Potočnik, Elementary abelian covers, JACO, 2004.

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$\Gamma = \Gamma_{g(x)}$ has vertex set $\mathbb{Z}_p^r \times \mathbb{Z}_n$ and $(\underline{v}, j) \sim (\underline{v} \pm \underline{u}_{j+1}, j+1)$

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$g(x) = (3 + 4x^2 + 2x^4 + x^6) \mid (x^8 - 1) \in \mathbb{Z}_5[x]$, not reflexible
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$\Gamma_{g(x)} = C4[200, 22]$ in Potočnik-Wilson census

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  $g_1(x)$ is reflexible and maximal weakly reflexible
  since $x^2 + 4x + 2$ and $x^2 + 2x + 4$ not reflexible.

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  since $x^2 + 4x + 2$ and $x^2 + 2x + 4$ not reflexible.
  So the cover is minimal and AT, $\Gamma = C4[147, 6]$.

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- type 1: $(x-1)^{k_1}(x+1)^{k_{-1}} \prod (x^2 - (a + a^{-1})x + 1)^{k_a}$
- type 2: $(x^2-1)^{k_{1,-1}} \prod (x^2 - (a - a^{-1})x - 1)^{k_a} (x-\theta)^{k_\theta}$
  $\theta^2 = -1$, $p \equiv 1 \mod 4$.

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Thank you!