Quasi-semiregular automorphisms of cubic and tetravalent arc-transitive graphs

István Kovács

University of Primorska, Slovenia

Joint work with Y.-Q. Feng, A. Hujdurović, K. Kutnar and D. Marušič

Graphs, groups, and more:

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Motivation, part 1

Let Γ be a finite undirected graph and let $G \leq \mathrm{Aut}\Gamma$.

Γ is ***G*-vertex-transitive** if G is transitive on the vertices.

A non-identity $g \in \mathrm{Aut}\Gamma$ is **semiregular** if the only power g^i fixing a vertex is the identity.

Polycirculant conjecture (Marušič)

Every vertex-transitive graph has a semiregular automorphism.

Remark: There is a slightly more general conjecture involving 2-cosed permutation groups due to M. Klin.

Remark: The conjecture does not hold for transitive permutation groups.

Motivation, part 1

Theorem (Marušič and Scapellato)

Every **cubic** *vertex-transitive graph has a semiregular automorphism.*

Theorem (Dobson, Malnič, Marušič and Nowitz)

Every **tetravalent** *vertex-transitive graph has a semiregular automorphism.*

Remark: The **fivevalent** case is still open.

Motivation, part 2

A transitive non-trivial permutation group G of a finite set Ω is a **Frobenius group** if every non-identity $g \in G$ fixes at most one point.

$$G = N \rtimes G_\omega, \text{ and } N \text{ is regular on } \Omega \text{ (Frobenius's theorem).}$$

A **graphical Frobenius representation** (GFR) of G is a graph Γ such that $\mathrm{Aut}\Gamma$ is permutation isomorphic to G (Doyle, Tucker and Watkins).

Example: The Paley graph $P(p)$ is a GFR for $\mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{p-1}{2}}$.

Quasi-semiregular automorphism

A permutation group G of a set Ω is **quasi-semiregular** if

- There exsits some $\omega \in \Omega$ fixed by any $g \in G$, and
- G is **semiregular** on $\Omega \setminus \{\omega\}$ (Kutnar, Malnič, Martínez and Marušič).

Equivalently:

A non-identity $g \in \mathrm{Aut}\Gamma$ is **quasi-semiregular** if

- g is not semiregular, and
- the only power g^i fixing two vertices is the identity.

Examples

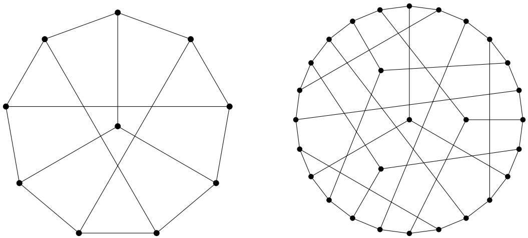

Figure : The Petersen graph and the Coxeter graph.

Examples

Let H be a group and $S \subset H$ such that

- $1_H \notin S$,
- $S = S^{-1} = \{s^{-1} : s \in S\}$.

The **Cayley graph** $\mathrm{Cay}(H, S) = (V, E)$, where

$$V = H \text{ and } E = \{(h, sh) : h \in H, \ s \in S\}.$$

If H is abelian and $|H|$ is odd, then

$$g : h \mapsto h^{-1} \ (h \in H)$$

is a quasi-semiregular automorpism of $\mathrm{Cay}(H, S)$.

s-arcs

Γ is G**-arc-transitive** if G is transitive on the arcs (= ordered pairs of adjacent vertices).

An s**-arc** of a graph Γ is a ordered $(s+1)$-tuple

$$(v_1, v_2, \ldots, v_{s+1})$$

such that $v_i \sim v_{i+1}$ and $v_i \neq v_{i+2}$.

Γ is (G, s)**-arc-transitive** (**regular**) if G is transitive (regular) on the s-arcs.

Our main result

Theorem (Feng, Hujdurović, K, Kutnar and Marušič)

Let Γ be a connected arc-transitive graph of valency $d \in \{3, 4\}$, and suppose that Γ admits a quasi-semiregular automorphism.

(i) *If $d = 3$, then Γ is isomorphic to K_4 or the Petersen graph or the Coxeter graph.*

(ii) *If $d = 4$ and Γ is 2-arc-transitive, then Γ is isomorphic to K_5.*

(iii) *If $d = 4$ and Γ is G-arc-transitive, where G is solvable and contains a quasi-semiregular automorphism, then Γ is isomorphic to $\mathrm{Cay}(A, X)$, where A is an abelian group of odd order and X is an orbit of a subgroup of $\mathrm{Aut}(A)$.*

Properties of quasi-semiregular automorphisms

For $N \lhd \mathrm{Aut}\Gamma$, **quotient graph** Γ_N has vertices the N-orbits, and edges (u^N, v^N) with $u^N \neq v^N$ and $(u, v) \in E\Gamma$.

If the mapping $V\Gamma \to V\Gamma_N, , v \mapsto v^N$ is locally bijective, then Γ is called the **normal cover** of Γ_N.

Lemma

Let Γ be a G-vertex-transitive graph, $N \lhd G$ a non-trivial normal semiregular subgroup and $1 < H \leq G$ a quasi-semiregular subgroup. Then

(i) *N is nilpotent, and if $|H|$ is even, then N is abelian and $G_v/C_{G_v}(N)$ has a non-trivial center.*

(ii) *If N is intransitive and Γ is a normal cover of Γ_N, then $HN/N \neq 1$ is quasi-semiregular on $V\Gamma_N$.*

Properties of quasi-semiregular automorphisms

Lemma

Let Γ be a G-vertex-transitive graph, and $H \leq G$ be a non-trivial subgroup which is quasi-semiregular on $V\Gamma$ with the fixed vertex v. Then $C_G(H) \leq N_G(H) \leq G_v$.

Proof.

Let $1 \neq h \in H$ and let $g \in N_G(H)$. Then $h^g \in H$, and thus v is the unique fixed vertex of h^g. On the other hand, h^g fixes v^g, and it follows that $g \in G_v$. □

Theorem (Tutte; Djoković and Miller)

If Γ is a cubic G-arc-transitive graph, then it is (G, s)-arc-regular for some $1 \leq s \leq 5$. Moreover, the structure of G_v is uniquely determined by s and is as in the Table below.

s	1	2	3	4	5
G_v	$\mathbb{Z}_3$	S_3	$\mathbb{Z}_2 \times S_3$	S_4	$\mathbb{Z}_2 \times S_4$

Table : Vertex-stabilisers in cubic s-arc-regular graphs.

Cubic arc-transitive graphs

Theorem (Feit and Thompson)

Let G be a finite group which contains a self-centralising subgroup of order 3. Then one of the following holds:

(i) $G \cong \mathrm{PSL}(2,7)$,

(ii) G *has a normal nilpotent subgroup* N *such that* $G/N \cong \mathbb{Z}_3$ *or* S_3,

(iii) G *has a normal* 2*-subgroup* N *such that* $G/N \cong A_5$.

Theorem (Morini)

Let G be a finite non-abelian simple group which contains a subgroup of order 3 whose centraliser in G is of order 6. Then $G \cong \mathrm{PSL}(2,11)$ or $\mathrm{PSL}(2,13)$.

Cubic arc-transitive graph

Γ is cubic (G, s)-regular, where

s	1	2	3	4	5
G_v	$\mathbb{Z}_3$	S_3	$\mathbb{Z}_2 \times S_3$	S_4	$\mathbb{Z}_2 \times S_4$

We prove that, if Γ has a quasi-semiregular automorphism, then it is also (H, s)-regular for some $s \in \{1, 2, 4\}$. Then we apply the Feit and Thomson's theorem:

(i) $H \cong \mathrm{PSL}(2, 7)$:
In this case Γ is isomorphic to the Coxeter graph.

(ii) H has a normal nilpotent subgroup N such that $H/N \cong \mathbb{Z}_3$ or S_3:
In this case Γ is isomorphic to K_4.

(iii) H has a normal 2-subgroup N such that $H/N \cong A_5$:
In this case Γ is isomorphic to the Petersen graph.

Tetravalent 2-arc-transitive graph

Observation: If Γ is a tetravalent graph having a quasi-semiregular automorphism, then $|V\Gamma|$ is odd.

If Γ is also G-vertex-transitive, then a Sylow 2-subgroup of G is contained in G_v.

Theorem

Let Γ be a tetravalent (G,s)-transitive graph of odd order. Then $s \leq 3$ and one of the following holds:

(i) G_v *is a* 2*-group for* $s = 1$.

(ii) $G_v \cong A_4$ *or* S_4 *for* $s = 2$.

(iii) $G_v \cong \mathbb{Z}_3 \times A_4$ *or* $\mathbb{Z}_3 \rtimes S_4$ *or* $S_3 \times S_4$ *for* $s = 3$.

Tetravalent 2-arc-transitive graph

Theorem (Malyushitsky)

Let T be a non-abelian simple group and let S be a Sylow 2-subgroup of G such that $|S| \leq 8$. Then, S, T and $Out(T)$ are given in the Table below.

S	T	Out(T)
$\mathbb{Z}_2^2$	$\mathrm{PSL}(2,4)$	$\mathbb{Z}_2$
	$\mathrm{PSL}(2,q)$, $q \equiv \pm 3 \pmod 8$	$\mathbb{Z}_2 \times \mathbb{Z}_d$, $q = p^d$
D_8	A_6	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	A_7	$\mathbb{Z}_2$
	$\mathrm{PSL}(2,7)$	S_3
	$\mathrm{PSL}(2,q)$, $q \equiv \pm 7 \pmod{16}$	$\mathbb{Z}_2 \times \mathbb{Z}_d$, $q = p^d$
$\mathbb{Z}_2^3$	J_1	trivial
	$PSL(2,8)$	$\mathbb{Z}_3$
	$R(3^{2n+1})$, $n > 1$	$\mathbb{Z}_{2n+1}$

Table : Non-abelian simple groups T with a Sylow 2-subgroup S of order 4 or 8.

Remark: The result is **CFSG**-free :)

Tetravalent 2-arc-transitive graph

Lemma

Let Γ be a tetravalent $(G, 2)$-arc-transitive graph, and suppose that G has a quasi-semiregular automorphism. If G is quasiprimitive on $V\Gamma$, then $\Gamma \cong K_5$ and $G \cong A_5$ or S_5.

We show that, if Γ is tetravalent $(G, 2)$-arc-transitive with a quasi-semiregular automorphism in G, then $G/O_{2'}(G)$ is quasiprimitive on $V\Gamma_{O_{2'}(G)}$. By the lemma $\Gamma_{O_{2'}(G)} \cong K_5$. Then we prove that $O_{2'}(G) = 1$.

Tetravalent arc-transitive graph (solvable case)

Lemma

Let Γ be a tetravalent G-arc-transitive graph such that $|V\Gamma| > 5$. Suppose that G contains a quasi-semiregular automorphism, and $N \lhd G$ is an intransitive minimal normal subgroup isomorphic to $\mathbb{Z}_p^n$ for some prime p. Then one of the following holds:

(i) *$N \cong \mathbb{Z}_p$ and G contains a regular normal subgroup L with $N \leq L$.*

(ii) *Γ is a normal cover of Γ_N.*

Remark: In the proof we use results of Gardiner and Praeger about tetravalent arc-transitive graphs.

Using the lemma, we show that, if Γ is tetravalent G-arc-transitive with a quasi-semiregular automorphism in G, then $O_{2'}(G)$ is regular and abelian, and by this

$$\Gamma \cong \mathrm{Cay}(O_{2'}(G), S) \text{ for some } S \subset O_{2'}(G).$$

Happy birthday Brian and Dragan!

$\sim$

Thank you for attention!