

On the isomorphisms of bi-Cayley graphs

Abstract

Let G be a group and S be a subset of G . Then $BCay(G, S)$, bi-Cayley graph of G with respect to S , is an undirected graph with vertex-set $G \times \{1, 2\}$ and edge-set $\{(g, 1), (sg, 2) \mid g \in G, s \in S\}$. For $\sigma \in \text{Aut}(G)$ and $g \in G$, we have $BCay(G, S) \cong BCay(G, gS^\sigma)$. A bi-Cayley graph $BCay(G, S)$ is called a *BCI-graph* if for any bi-Cayley graph $BCay(G, T)$, whenever $BCay(G, S) \cong BCay(G, T)$ we have $T = gS^\sigma$ for some $g \in G$ and $\sigma \in \text{Aut}(G)$. A group G is called a *BCI-group* if every bi-Cayley graph of G is a *BCI-graph*. In this lecture, we discuss recent results and future directions of classifying finite *BCI*-groups.

Introduction and results

A fundamental problem about Cayley graphs is the so called isomorphism problem, that is, given two Cayley graphs $Cay(G, S)$ and $Cay(H, T)$ determine whether or not $Cay(G, S) \cong Cay(H, T)$. It follows quickly from the definition that for any automorphism $\alpha \in \text{Aut}(G)$, the graphs $Cay(G, S)$ and $Cay(G, S^\alpha)$ are isomorphic, namely, α induces an isomorphism between these graphs. Such an isomorphism is also called a *Cayley isomorphism*.

In 1967, Adam [1] conjectured that two Cayley graphs over the cyclic group are isomorphic if and only if there is a Cayley isomorphism which maps one to the other. Soon afterwards, Elspas and Turner [4] found the counterexample for $n = 8$. This also motivated the following definition. A Cayley graph $Cay(G, S)$ has the *CI-property* (for short, it is a *CI-graph*) if for any Cayley graph $Cay(G, T)$, $Cay(G, S) \cong Cay(G, T)$ implies that $T = S^\alpha$ for some $\alpha \in \text{Aut}(G)$. Finite *CI*-groups have attracted considerable attention over the last 50 years. The problem of classifying finite *CI*-groups is still open.

In 2008, motivated by the concepts *CI-graph*, m - *BCI-group* and *CI-group*, Xu et al. [13] introduced the concepts *BCI-graph*, m - *BCI-group* and *BCI-group*, respectively. We say that a bi-Cayley graph $BCay(G, S)$ is a *BCI-graph* if whenever $BCay(G, S) \cong BCay(G, T)$ for some subset T of G , the set $T = gS^\alpha$ for some $g \in G$ and automorphism $\alpha \in \text{Aut}(G)$. The group G is an m - *BCI-group* if every bi-Cayley graph over G of valency at most m is a *BCI-graph*, and G is a *BCI-group* if every bi-Cayley graph over G is a *BCI-graph*. The theory of *BCI*-graphs and *BCI*-groups is less developed as in the case of *CI*-graphs and *CI*-groups. Several basic properties have been obtained by Jin and Liu in a series of papers [5-8], also by Koike et.al. [9-12] and very recently, by the present author [2-3]. We will discuss the relation between *BCI*-groups and *CI*-groups. In fact, our primary motivation by studying *BCI*-graphs and *BCI*-groups is that these objects can bring new insight into the old problem of classifying *CI*-groups. In [2] it is conjectured that every *BCI-group* is a *CI-group* and it is proved that every group of prime order is a *BCI-group* and every Sylow subgroup of a *BCI-group* is elementary abelian. Also, in [3], it is proved that every *BCI-group* is solvable.

Since every bi-Cayley graph over an abelian group is a Cayley graph over a generalized dihedral group, it seems that classifying finite abelian *BCI*-groups can help to classifying generalized dihedral *CI*-groups. In this lecture, we present some new results about classifying finite abelian *BCI*-groups.

This is a joint work with Majid Arezoomandb.

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