Classification of vertex-transitive digraphs via automorphism group

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This is joint work with Ademir Hujdurović, Klavdija Kutnar, Joy Morris, and Primŏz Potŏcnik.

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8. Cheryl E. Praeger and Ming Yao Xu, *Vertex-primitive graphs of order a product of two distinct primes*, J. Combin. Theory Ser. B **59** (1993), no. 2, 245–266. MR1244933 (94j:05061)

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Over the years, it has become apparent that there are some small mistakes and small "gaps" in these classifications. Our goal is to fix all known errors (some of which have propagated in the literature) and to fill in the "gaps" that we can see.

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A “gap” is filled by the following result:

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Theorem

Let Γ be a vertex-transitive digraph of order pq, where $q < p$ are distinct primes such that $\mathrm{Aut}(\Gamma)$ is a quasiprimitive almost simple group that admits an $\mathrm{Aut}(\Gamma)$-invariant partition with blocks of size q, and Γ is a (q, p)-metacirculant. Then one of the following is true:

1. *Γ is isomorphic to a generalized orbital digraph of $\mathrm{PSL}(2, 11)$ that is not a generalized orbital digraph of $\mathrm{PGL}(2, 11)$ of order 55. Moreover, Γ is a Cayley digraph of the nonabelian group of order 55, and its full automorphism group is $\mathrm{PSL}(2, 11)$, or*
2. *Γ is isomorphic to a generalized orbital digraph of $\mathrm{PSL}(3, 2)$ of order 21 that is not a generalized orbital digraph of $\mathrm{P\Gamma L}(3, 2)$. Moreover, Γ is a Cayley digraph of the nonabelian group of order 21, and its full automorphism group is $\mathrm{PSL}(3, 2)$.*

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Theorem

Let $p = 2^s + 1$ be a Fermat prime and $q|(2^s - 1)$ be prime. Let $\Gamma = X(2^s, q, S, T)$ be a symmetric Marušič-Scapellato digraph and assume that $\mathrm{SL}(2, 2^s) \leq \mathrm{Aut}(\Gamma) \leq \Sigma\mathrm{L}(2, 2^s)$. Let a be the order of 2 modulo q. Then $S = \emptyset$ and one of the following is true:

1. *$T = \{0\}$, Γ has valency q, and automorphism group $\Sigma\mathrm{L}(2, 2^s)$.*
2. *There is a divisor b of $\gcd(a, s)$ and $1 < a/b < q - 1$ such that $T = U_{b,i} = \{i2^{bj} : 0 \leq j < a/b\}$. There are exactly $(q-1)/a$ distinct graphs of this type for a given b, each of valency qa/b, and the automorphism group of each is $\langle \mathrm{SL}(2, 2^s), L\rangle$ where $L \leq \langle f \rangle$ is of order s/b. Up to isomorphism, there are exactly $(q-1)/b$ such graphs.*

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Theorem

Let $p = 2^s + 1$ be a Fermat prime, $q|(2^s - 1)$ be prime, and Γ be a Marušič-Scapellato digraph of order qp. Then Γ or its complement is $X(2^s, q, S, T)$ and one of the following is true.

1. $\mathrm{Aut}(\Gamma)$ *is primitive and*
   1. $s = 2$, $qp = 15$, $S = \mathbb{Z}_3^*$ *and* $T = \{0\}, \{1\}$, *or* $\{2\}$. *Then* Γ *is isomorphic to the line graph of* K_6 *and has automorphism group* $d^{-1}\Sigma\mathrm{L}(2,4)d \cong S_6$ *for some* $d \in Z$.
   2. $p = k^2 + 1$, $q = k + 1$, $S = \mathbb{Z}_q^*$ *and* $|T| = 1$. *Then there exists* $d \in Z/\mathcal{D}_\ell$ *such that* $\mathrm{Aut}(\Gamma) = d^{-1}\mathrm{P\Gamma Sp}(4,k)d$.
   3. $S = \mathbb{Z}_q^*$, $T = \mathbb{Z}_q$, *and* Γ *is a complete graph with automorphism group* S_{qp}.
2. $\mathrm{Aut}(\Gamma)$ *is imprimitive and*
   1. $S < \mathbb{Z}_q^*$, $T = \mathbb{Z}_q$, Γ *is degenerate, and* $\mathrm{Aut}(\Gamma) \cong S_p \wr \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_q, S))$.
   2. *In all other cases there exists* $L \le \langle f/\mathcal{D}_\ell\rangle$ *and* $d \in Z/\mathcal{D}_\ell$ *such that*

$$\mathrm{Aut}(\Gamma) = d^{-1}\langle \mathrm{SL}(2, 2^s), L\rangle d$$

HAPPY BIRTHDAYS!