MAJORITY COLORING GAMES

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Graphs, groups and more:
celebrating Brian Alspach's 80th and Dragan Marušič's 65th birthdays

28-05-2018, Koper

Definition 1

Let G be a simple graph. A coloring c of the vertices of G is called a *majority coloring*, if for every vertex v, the number of its neighbors in color $c(v)$ is at most $\frac{1}{2}d(v)$, where $d(v)$ denotes the number of neighbors of vertex v in G.

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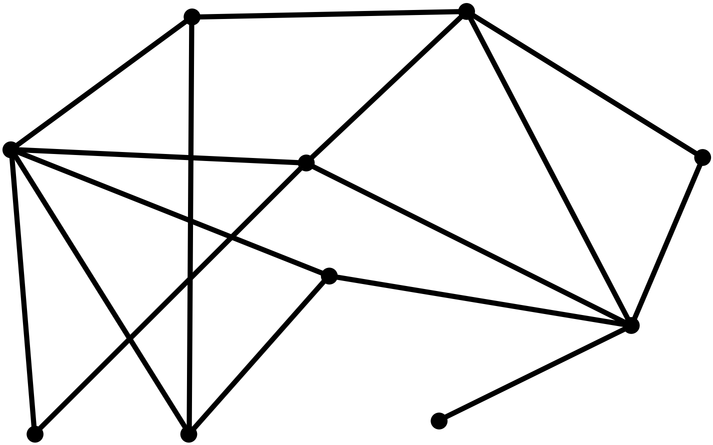
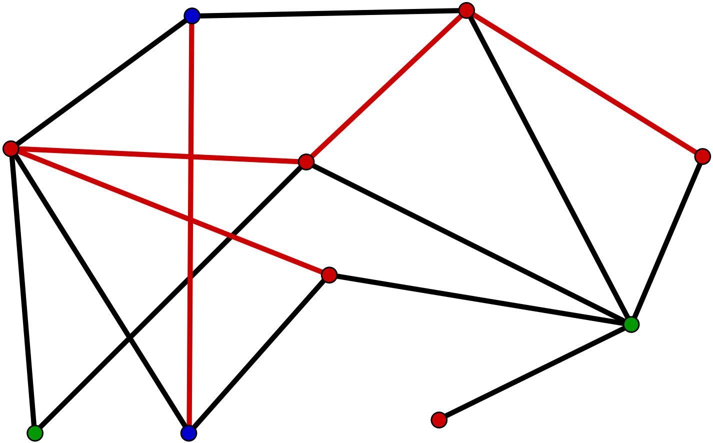
Definition 2

The *majority chromatic number* of a graph G is the least number of colors in a majority coloring of G. We denote it by $\mu(G)$.

Example of majority coloring



Example of majority coloring



Theorem 3

For every finite graph G exists majority coloring using only two colors.

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Consider 2-coloring minimazing the number of monochromatic edges.

Majority coloring game

Two players

ALICE
BOB

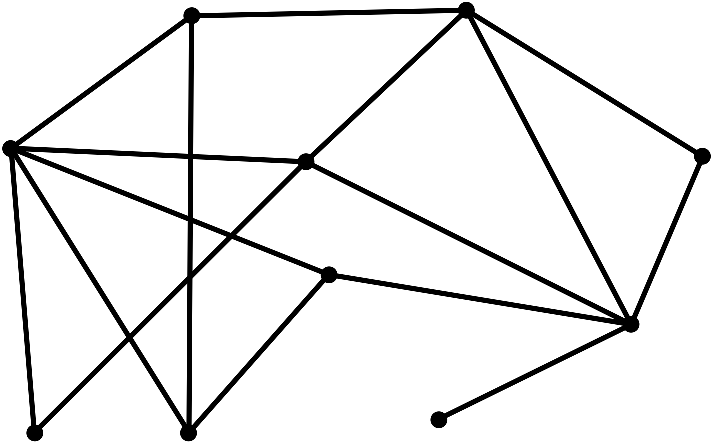
Majority coloring game

Two players are alternately coloring vertices of considered graph using colors from given color set C. After each move the majority rule should be satisfied for all vertices.

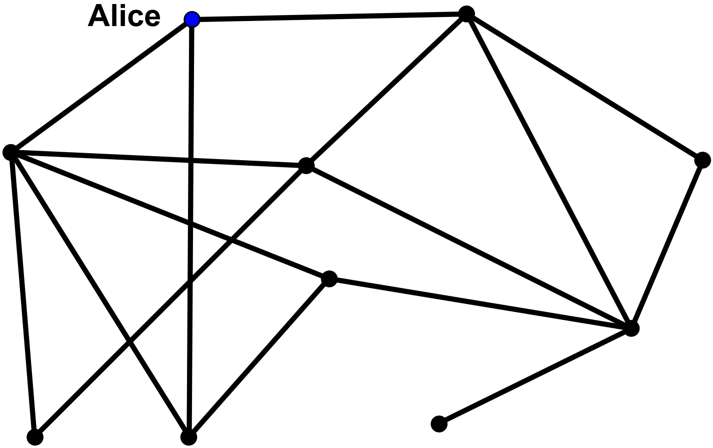
ALICE wants to create majority coloring of the whole graph.

BOB wants to create situation where exists vertex v which cannot be colored with any color from the set C without breaking the majority rule.

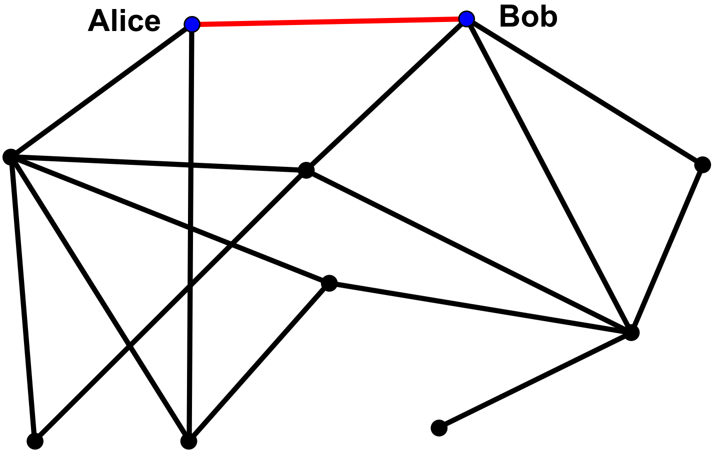
Example of majority coloring game



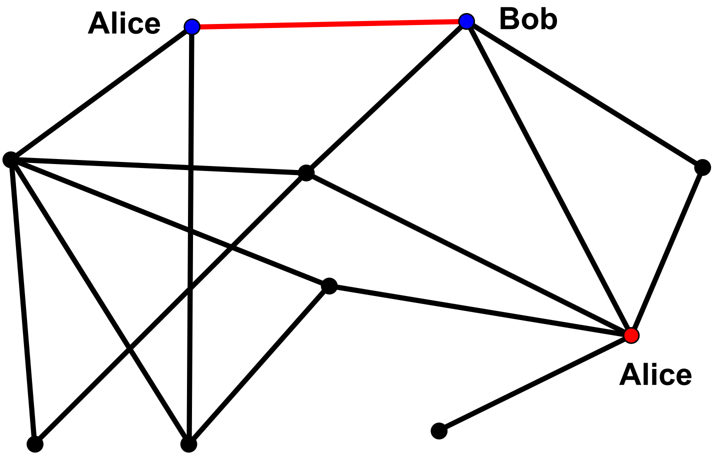
Example of majority coloring game



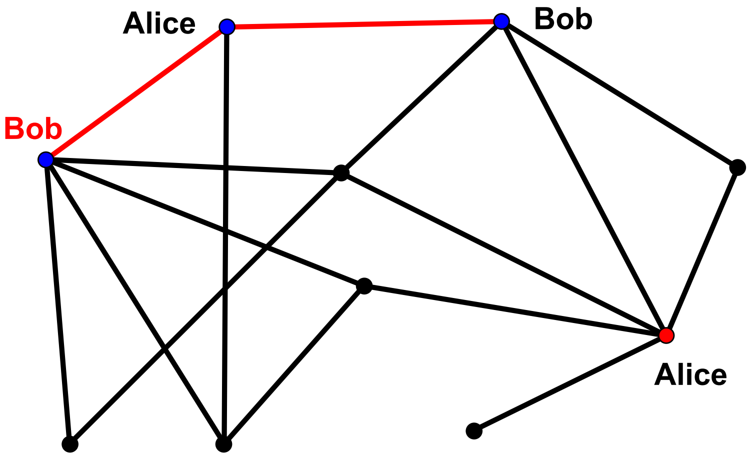
Example of majority coloring game



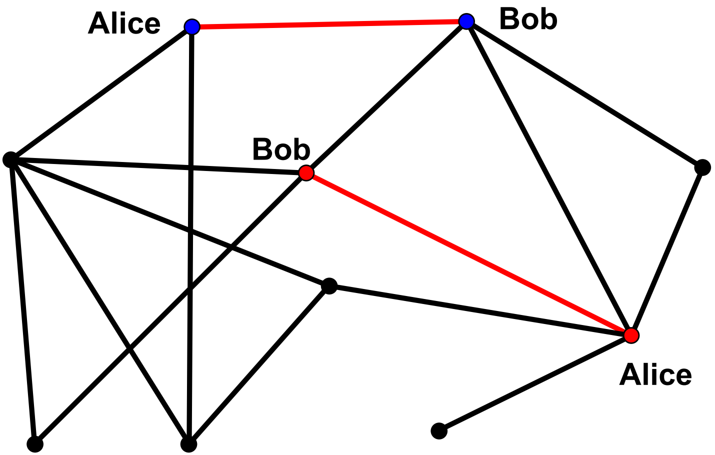
Example of majority coloring game



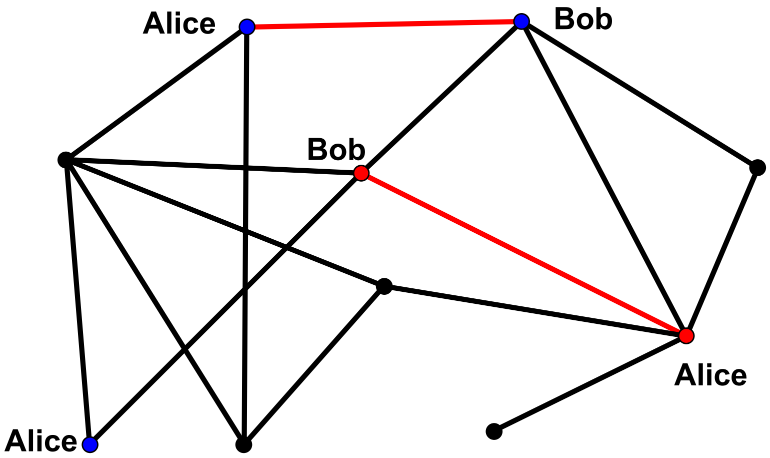
Example of majority coloring game



Example of majority coloring game



Example of majority coloring game



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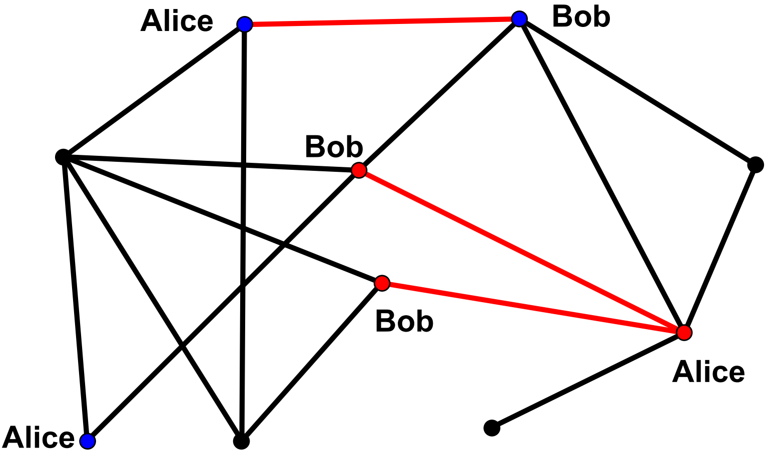

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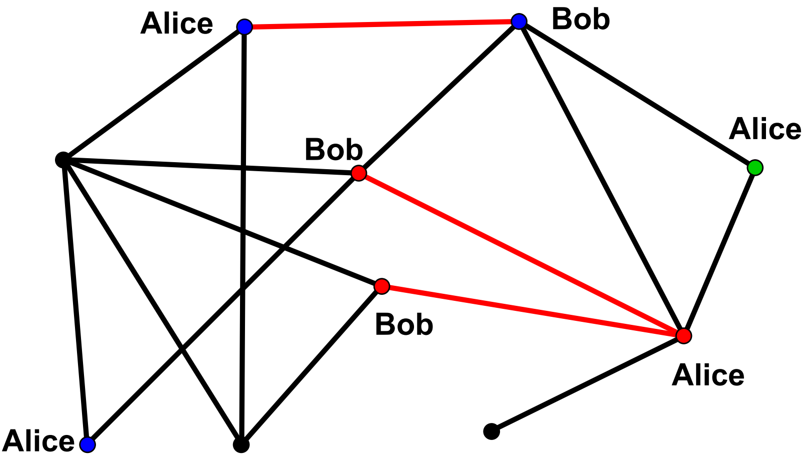

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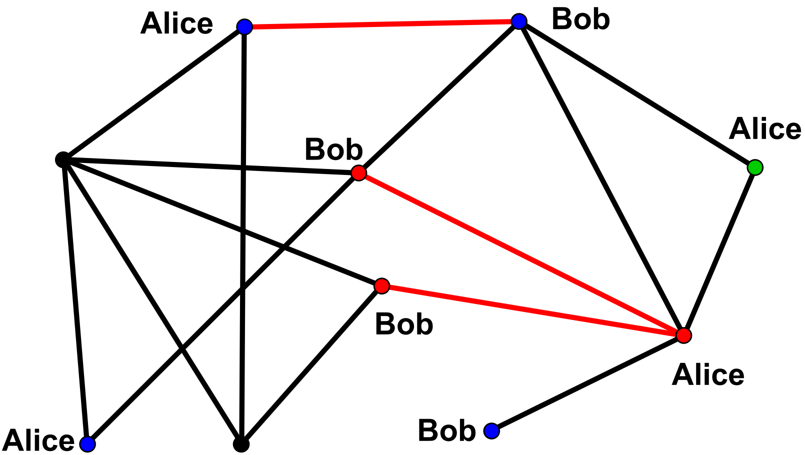

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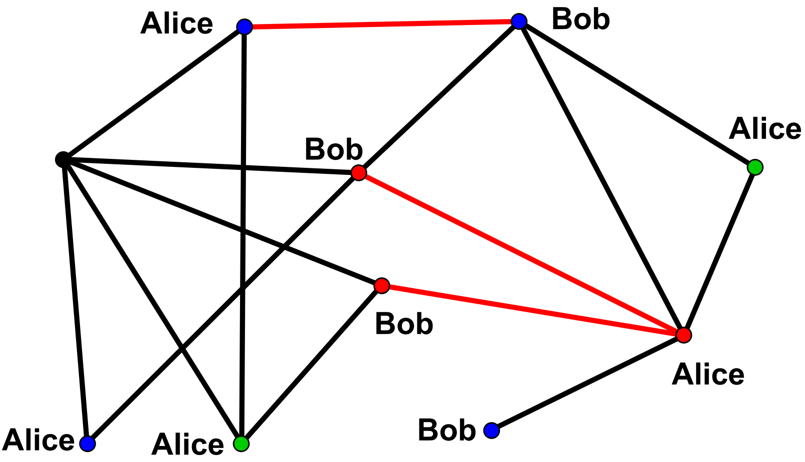

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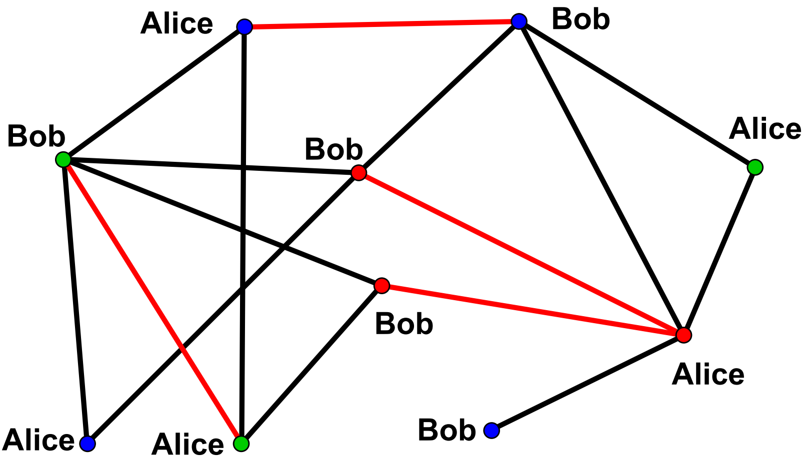

The least number of colors for which Alice has winning strategy on graph G is called the *majority game chromatic number* and denoted by $\mu_g(G)$

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Theorem 4

For every positive integer n there exists a bipartite graph $G(n)$ such that $\mu_g(G(n)) > n$.

Construction of the $(p, q) - massif$

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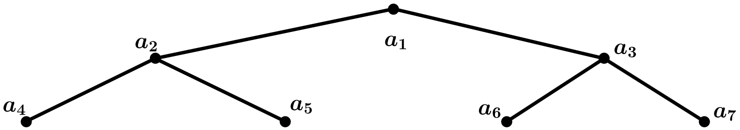

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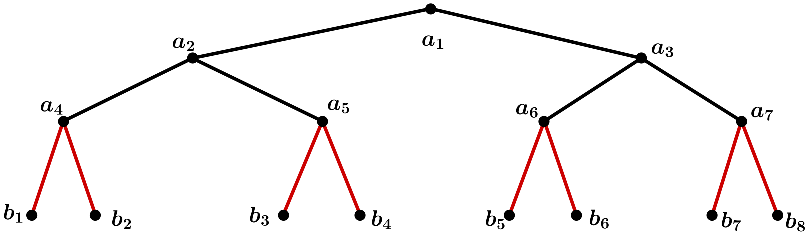

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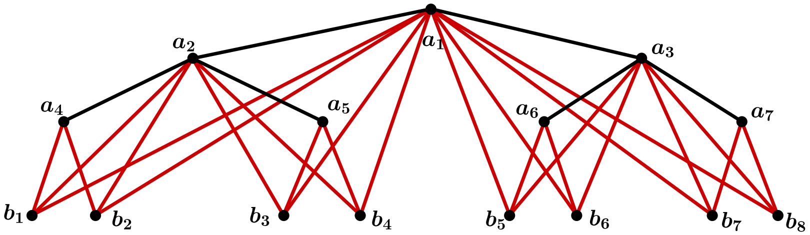

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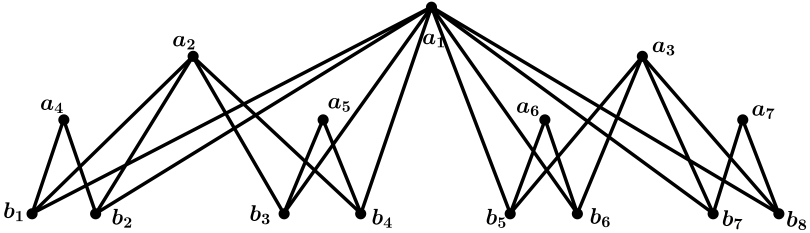

The $(3, 2)$-massif

Construction of the graph $G(3)$

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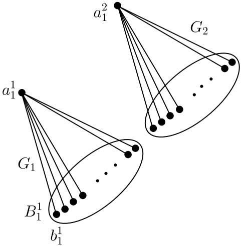
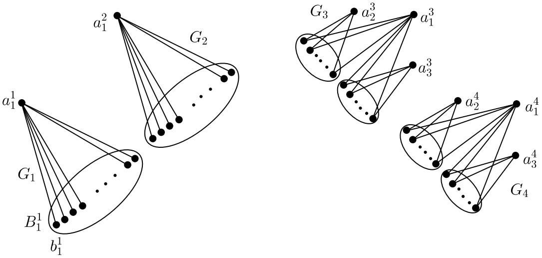
- two copies of the $(1, 2^k)$-massif – parts G_1, G_2


a_1^2
G_2
a_1^1
G_1
B_1^1
b_1^1

Construction of the graph $G(3)$

- two copies of the $(1, 2^k)$-massif – parts G_1, G_2
- two copies of the $(2, 2^{k-1})$-massif – parts G_3, G_4


a_1^2
G_2
a_1^1
G_1
B_1^1
b_1^1
G_3
a_2^3
a_1^3
a_3^3
a_2^4
a_1^4
a_3^4
G_4

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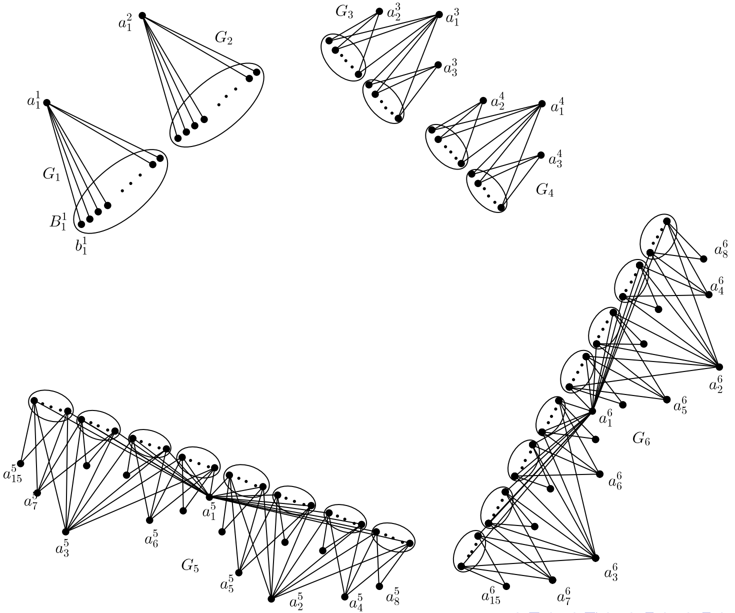
- two copies of the $(1, 2^k)$-massif – parts G_1, G_2
- two copies of the $(2, 2^{k-1})$-massif – parts G_3, G_4
- two copies of the $(4, 2^{k-2})$-massif – parts G_5, G_6


a_1^2
G_2
a_1^1
G_1
B_1^1
b_1^1
G_3
a_2^3
a_1^3
a_3^3
a_2^4
a_1^4
a_3^4
G_4
a_8^6
a_4^6
a_2^6
a_5^6
a_1^6
G_6
a_6^6
a_3^6
a_{15}^6
a_7^6
a_{15}^5
a_7^5
a_3^5
a_6^5
a_1^5
G_5
a_5^5
a_2^5
a_4^5
a_8^5

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In general case for every color $i \in [n]$ there are exactly two copies of the $(2^{i-1}, 2^{k-i+1})$-massif G_{2i-1} and G_{2i}.

Construction of the graph $G(3)$

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- two copies of the $(2, 2^{k-1})$-massif – parts G_3, G_4
- two copies of the $(4, 2^{k-2})$-massif – parts G_5, G_6
- independent set C of 2^k vertices $c_1, \ldots, c_{2^k}$

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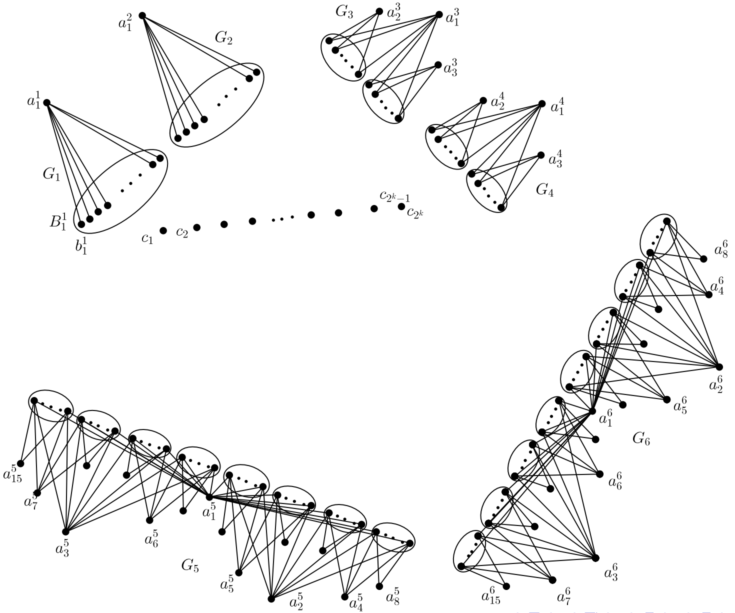
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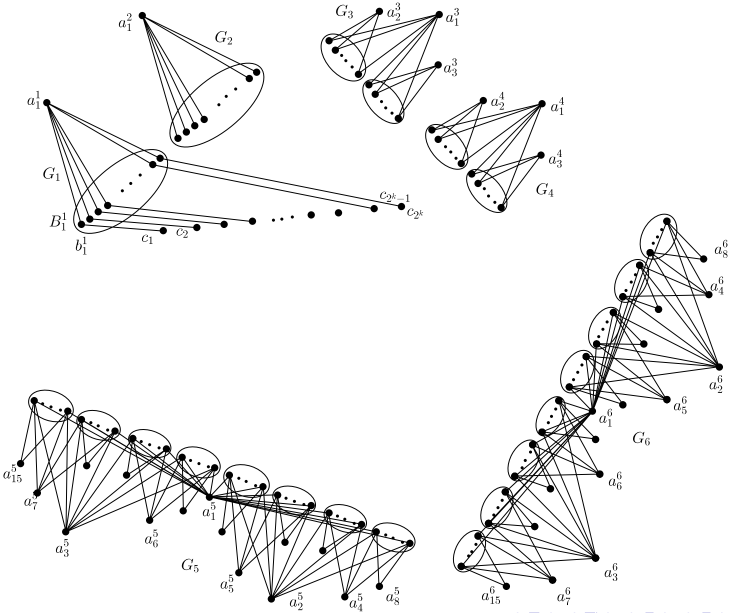
G_2
G_3
a_2^3
a_1^3
a_3^3
a_1^1
a_2^4
a_1^4
G_1
a_3^4
G_4
B_1^1
b_1^1
c_1
c_2
c_{2^k-1}
c_{2^k}
a_8^6
a_4^6
a_2^6
a_5^6
a_1^6
G_6
a_6^6
a_3^6
a_{15}^6
a_7^6
a_{15}^5
a_7^5
a_3^5
a_6^5
a_1^5
G_5
a_5^5
a_2^5
a_4^5
a_8^5


G_1
G_2
G_3
G_4
G_5
G_6
a_1^1
a_1^2
a_1^3
a_2^3
a_3^3
a_1^4
a_2^4
a_3^4
B_1^1
b_1^1
c_1
c_2
c_{2^k-1}
c_{2^k}
a_1^5
a_2^5
a_3^5
a_4^5
a_5^5
a_6^5
a_7^5
a_8^5
a_{15}^5
a_1^6
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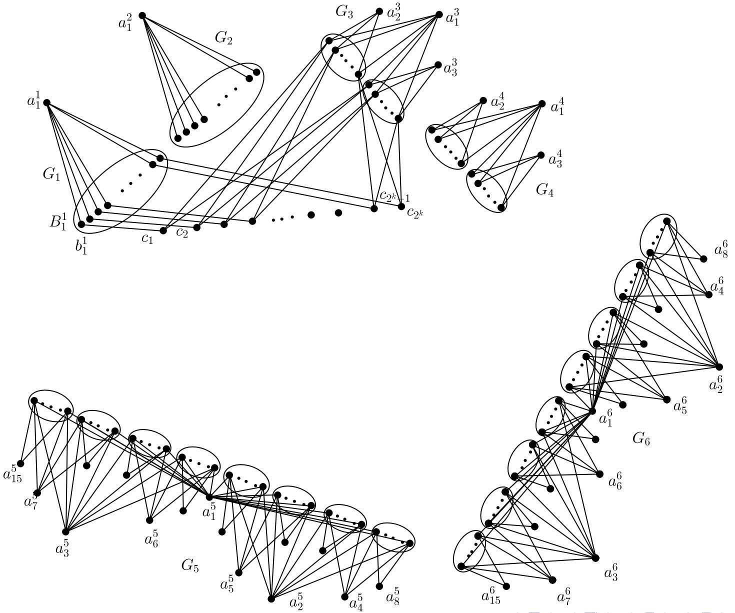
a_8^6
a_{15}^6


a_1^2
G_2
G_3
a_2^3
a_1^3
a_3^3
a_1^1
a_2^4
a_1^4
G_1
a_3^4
G_4
c_{2^k-1}
c_{2^k}
B_1^1
c_1
c_2
b_1^1
a_8^6
a_4^6
a_2^6
a_5^6
a_1^6
G_6
a_6^6
a_3^6
a_{15}^6
a_7^6
a_{15}^5
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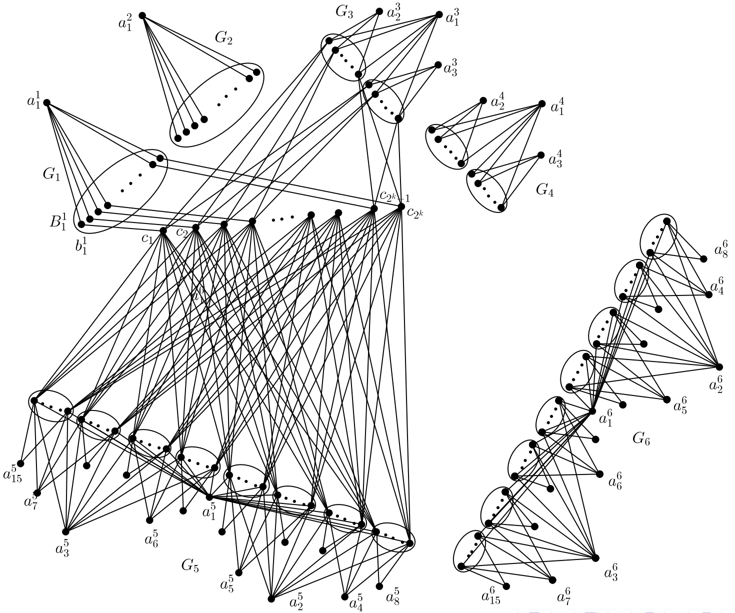
a_4^5
a_8^5


G_1
G_2
G_3
G_4
G_5
G_6
a_1^1
a_1^2
a_1^3
a_2^3
a_3^3
a_1^4
a_2^4
a_3^4
B_1^1
b_1^1
c_1
c_2
c_{2^k+1}
c_{2^k}
a_1^5
a_2^5
a_3^5
a_4^5
a_5^5
a_6^5
a_7^5
a_8^5
a_{15}^5
a_1^6
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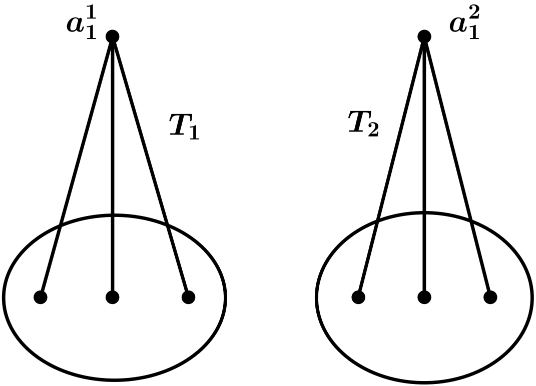
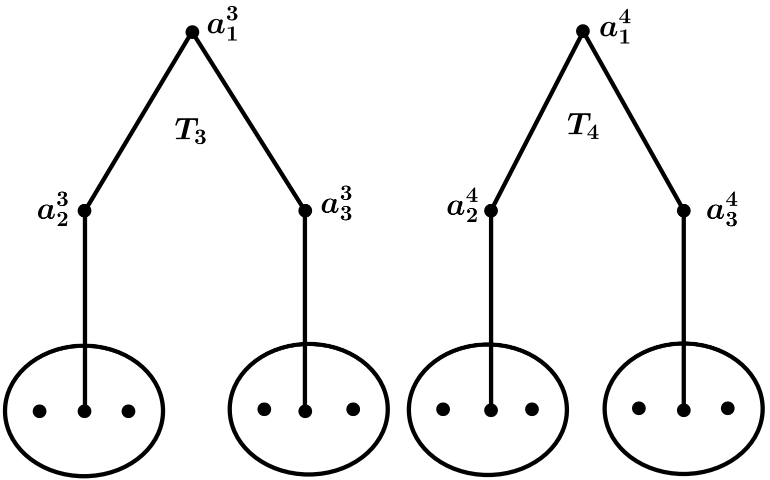
a_8^6
a_{15}^6


a_1^1
T_1
a_1^2
T_2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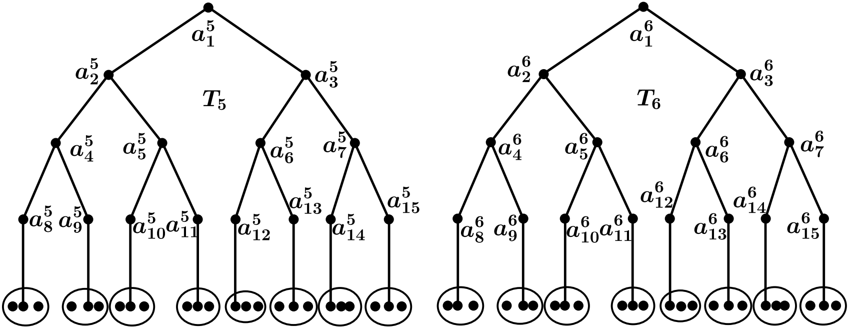
T_3
a_2^3
a_3^3
a_1^4
T_4
a_2^4
a_3^4


T_5
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a_3^5
a_4^5
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a_6^5
a_7^5
a_8^5
a_9^5
a_{10}^5
a_{11}^5
a_{12}^5
a_{13}^5
a_{14}^5
a_{15}^5
T_6
a_1^6
a_2^6
a_3^6
a_4^6
a_5^6
a_6^6
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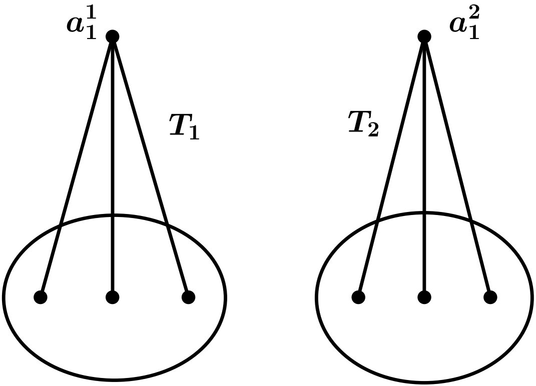
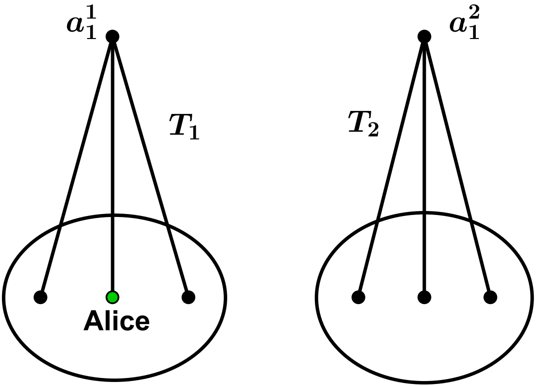
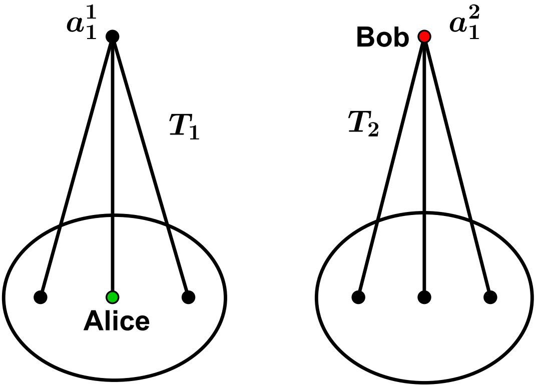
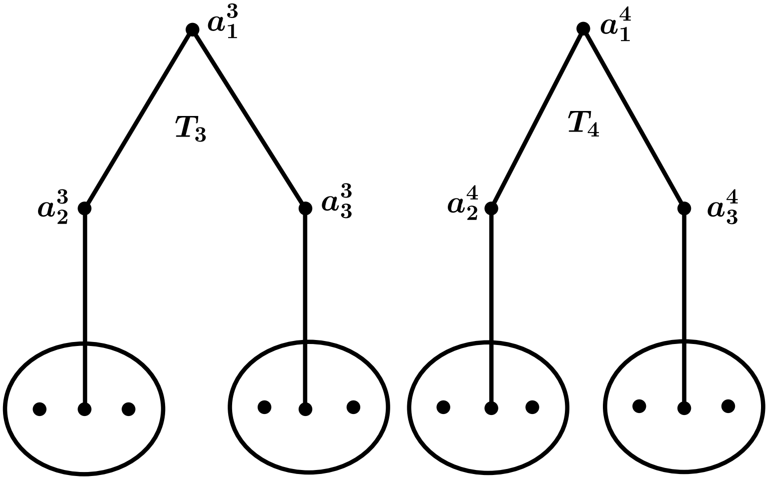
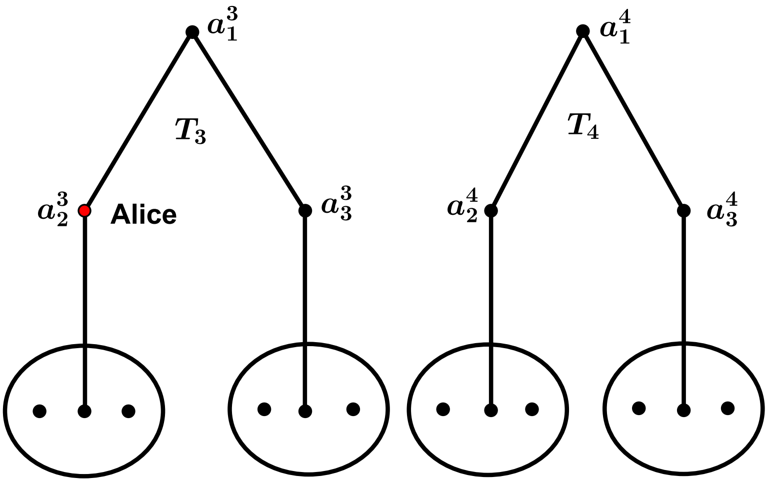
a_{14}^6
a_{15}^6


a_1^1
T_1
T_2
a_1^2


a_1^1
T_1
Alice
a_1^2
T_2


a_1^1
T_1
Alice
Bob
a_1^2
T_2


a_1^3
T_3
a_2^3
a_3^3
a_1^4
T_4
a_2^4
a_3^4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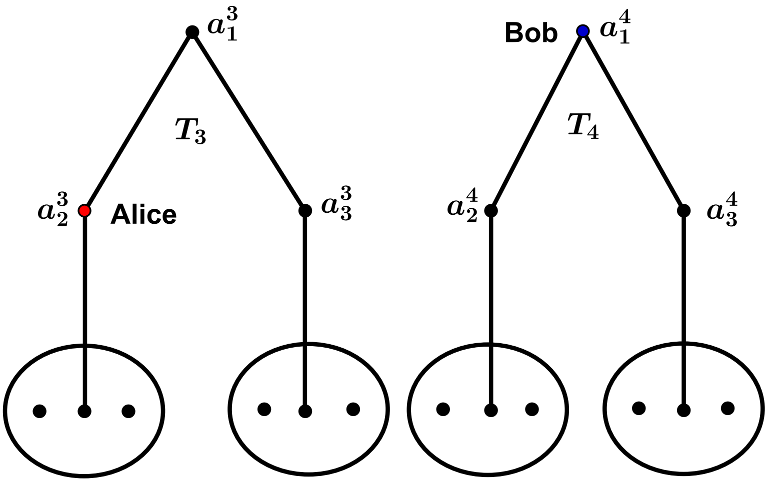
T_3
a_2^3
Alice
a_3^3
a_1^4
T_4
a_2^4
a_3^4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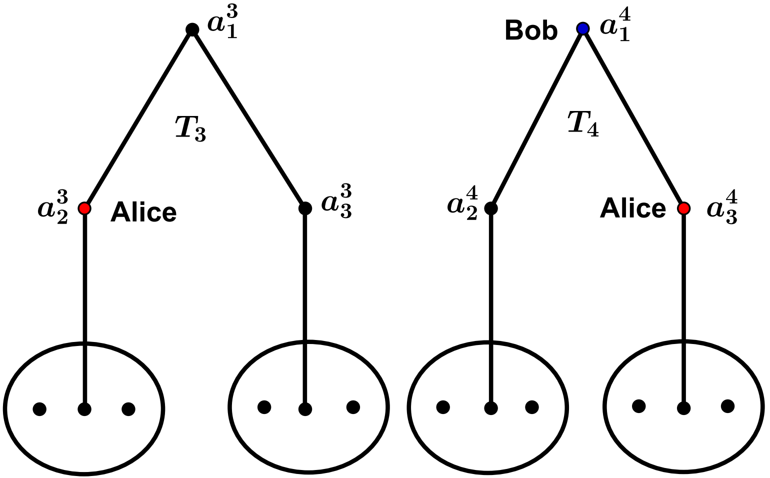
a_2^3
Alice
a_3^3
Bob
a_1^4
T_4
a_2^4
a_3^4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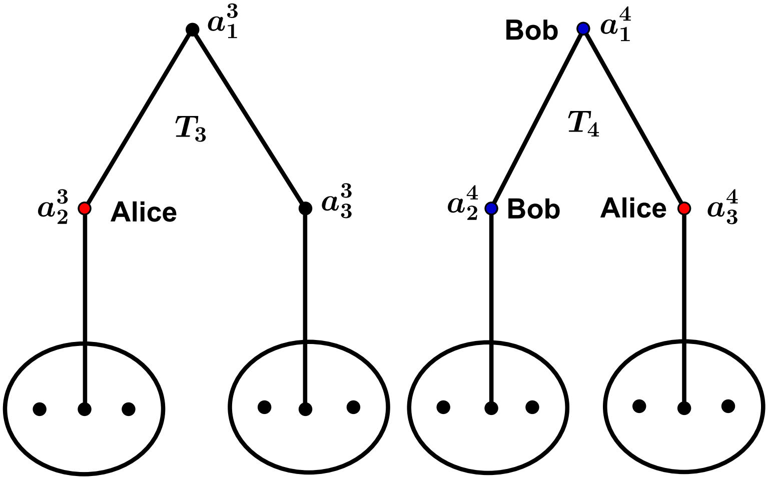
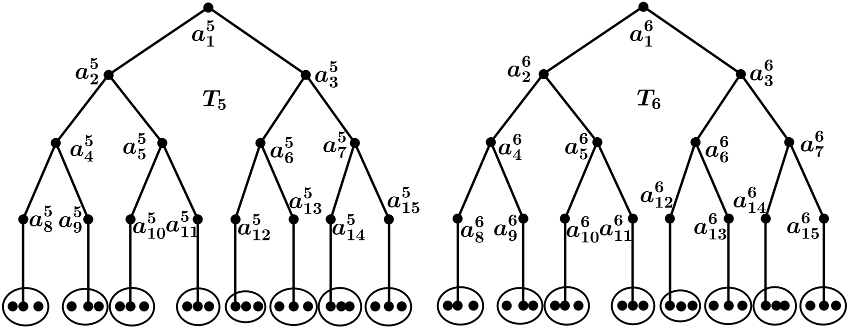
T_3
a_2^3
Alice
a_3^3
Bob
a_1^4
T_4
a_2^4
Alice
a_3^4


a_1^3
T_3
a_2^3
Alice
a_3^3
Bob
a_1^4
T_4
a_2^4
Bob
Alice
a_3^4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a_2^5
a_3^5
T_5
a_4^5
a_5^5
a_6^5
a_7^5
a_8^5
a_9^5
a_{10}^5
a_{11}^5
a_{12}^5
a_{13}^5
a_{14}^5
a_{15}^5
a_1^6
a_2^6
a_3^6
T_6
a_4^6
a_5^6
a_6^6
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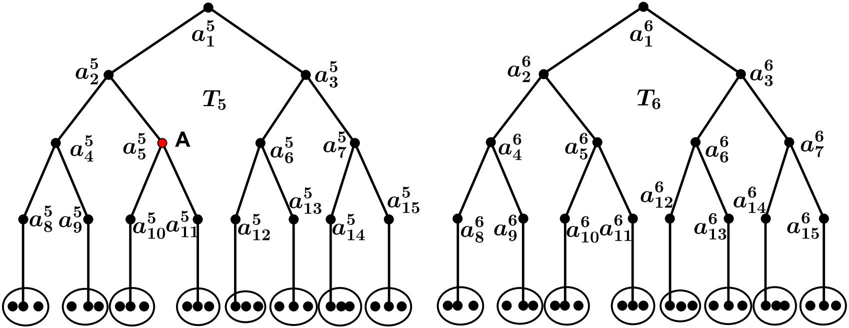
a_{14}^6
a_{15}^6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a_3^5
T_5
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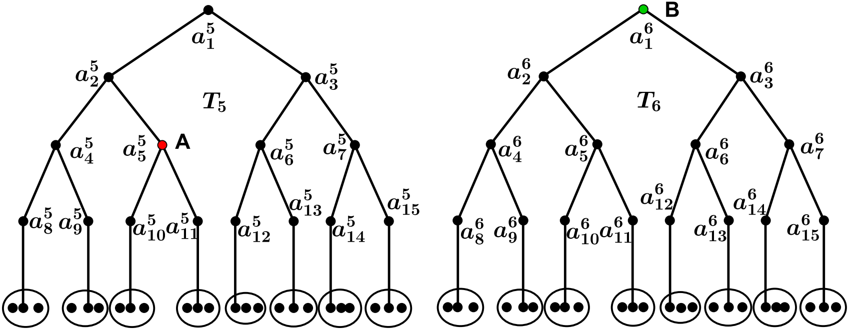
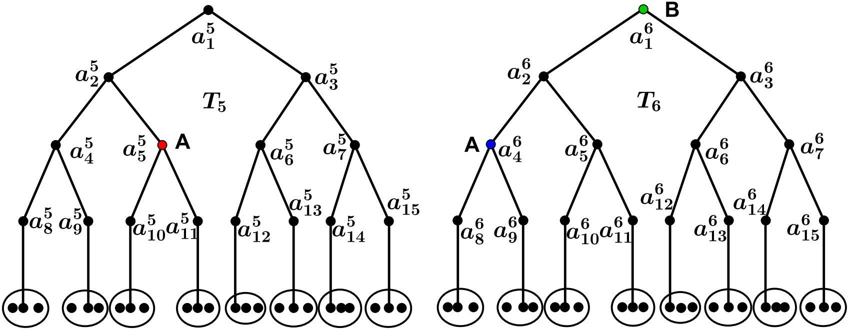
a_5^5
A
a_6^5
a_7^5
a_8^5 a_9^5 a_{10}^5 a_{11}^5 a_{12}^5 a_{13}^5 a_{14}^5 a_{15}^5
a_1^6
a_2^6
a_3^6
T_6
a_4^6 a_5^6 a_6^6 a_7^6
a_8^6 a_9^6 a_{10}^6 a_{11}^6 a_{12}^6 a_{13}^6 a_{14}^6 a_{15}^6


B
a_1^5
a_2^5
a_3^5
T_5
A
a_4^5
a_5^5
a_6^5
a_7^5
a_8^5 a_9^5 a_{10}^5 a_{11}^5 a_{12}^5 a_{13}^5 a_{14}^5 a_{15}^5
a_1^6
a_2^6
a_3^6
T_6
a_4^6
a_5^6
a_6^6
a_7^6
a_8^6 a_9^6 a_{10}^6 a_{11}^6 a_{12}^6 a_{13}^6 a_{14}^6 a_{15}^6


a_1^5
a_2^5
a_3^5
T_5
a_4^5
a_5^5
A
a_6^5
a_7^5
a_8^5
a_9^5
a_{10}^5
a_{11}^5
a_{12}^5
a_{13}^5
a_{14}^5
a_{15}^5
B
a_1^6
a_2^6
a_3^6
T_6
A
a_4^6
a_5^6
a_6^6
a_7^6
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a_9^6
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a_{12}^6
a_{13}^6
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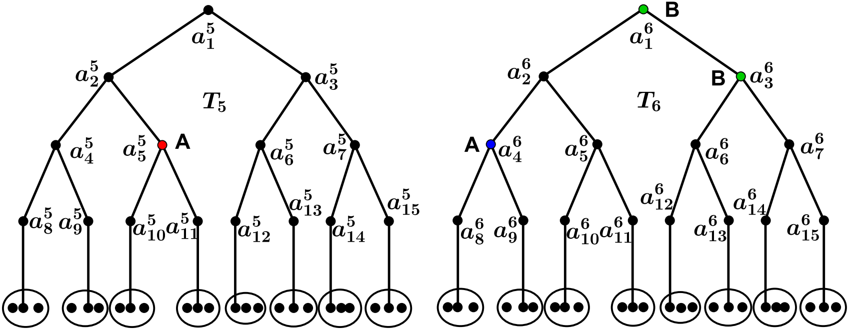
a_{15}^6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A
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a_7^5
a_8^5 a_9^5
a_{10}^5 a_{11}^5
a_{12}^5
a_{13}^5
a_{14}^5
a_{15}^5
B
a_1^6
a_2^6
B
a_3^6
T_6
A
a_4^6
a_5^6
a_6^6
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a_{10}^6 a_{11}^6
a_{12}^6
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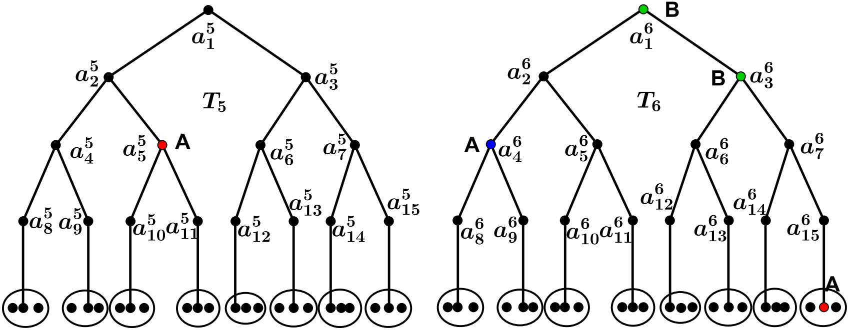
a_{15}^6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A
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$a_{10}^5 a_{11}^5$
a_{12}^5
a_{13}^5
a_{14}^5
a_{15}^5
B
a_1^6
a_2^6
B
a_3^6
T_6
A
a_4^6
a_5^6
a_6^6
a_7^6
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$a_{10}^6 a_{11}^6$
a_{12}^6
a_{13}^6
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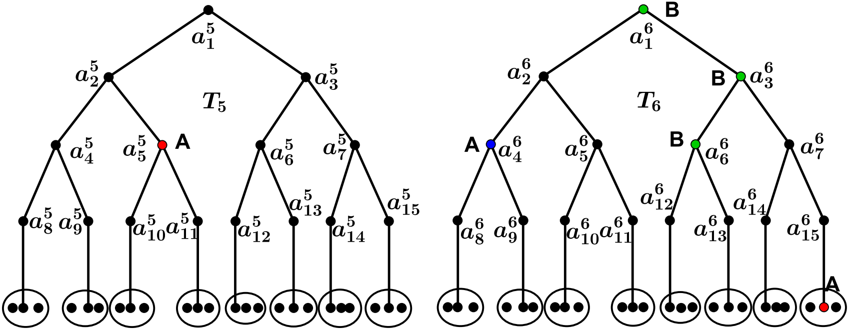
a_{15}^6
A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a_{12}^5
a_{13}^5
a_{14}^5
a_{15}^5
B
a_1^6
a_2^6
B
a_3^6
T_6
A
a_4^6
a_5^6
B
a_6^6
a_7^6
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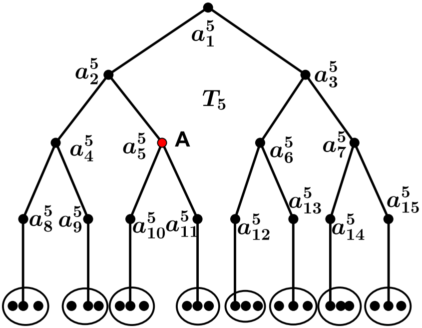
a_{15}^6
A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a_{10}^5
a_{11}^5
a_{12}^5
a_{13}^5
a_{14}^5
a_{15}^5
B
a_1^6
a_2^6
B
a_3^6
T_6
A
a_4^6
a_5^6
B
a_6^6
a_7^6
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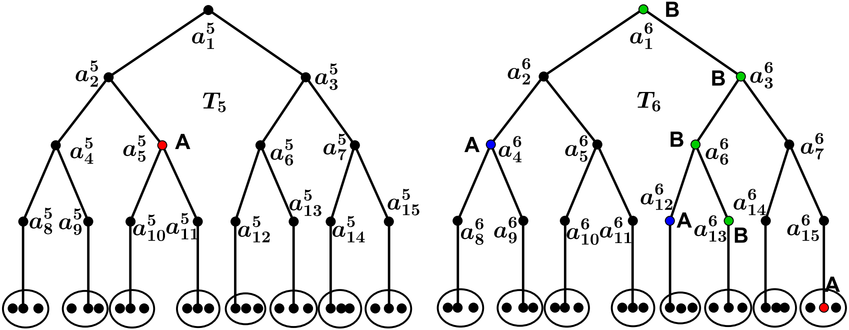
A
a_{13}^6
a_{14}^6
a_{15}^6
A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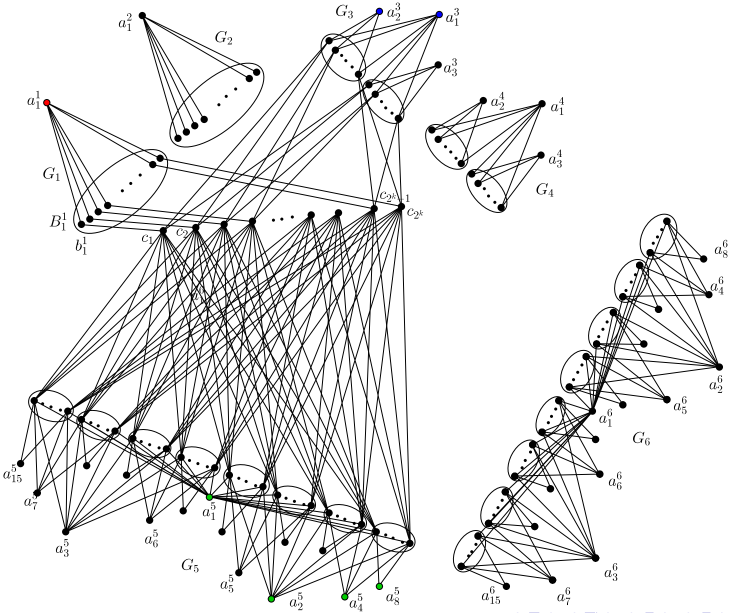
a_2^5
a_3^5
a_4^5
a_5^5
A
a_6^5
a_7^5
a_8^5
a_9^5
a_{10}^5
a_{11}^5
a_{12}^5
a_{13}^5
a_{14}^5
a_{15}^5
T_6
B
a_1^6
a_2^6
B
a_3^6
A
a_4^6
a_5^6
B
a_6^6
a_7^6
a_8^6
a_9^6
a_{10}^6
a_{11}^6
a_{12}^6
A
a_{13}^6
B
a_{14}^6
a_{15}^6
A


a_1^2
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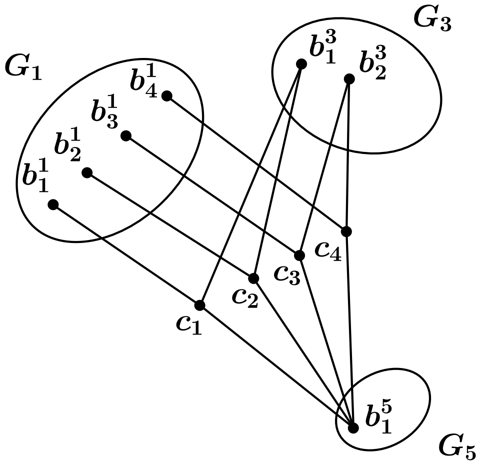
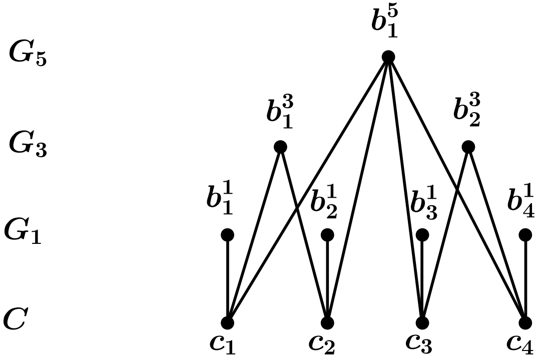
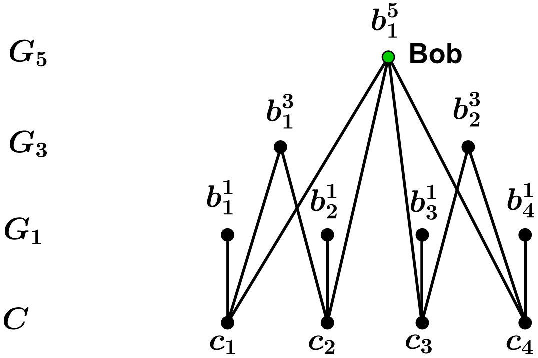
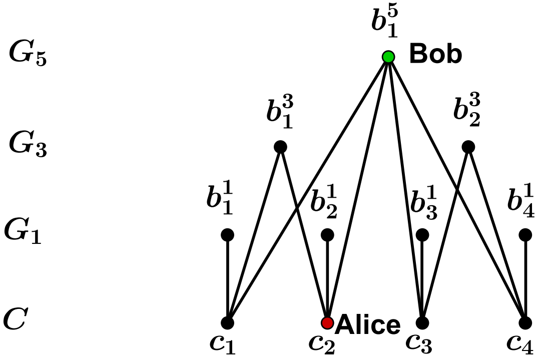
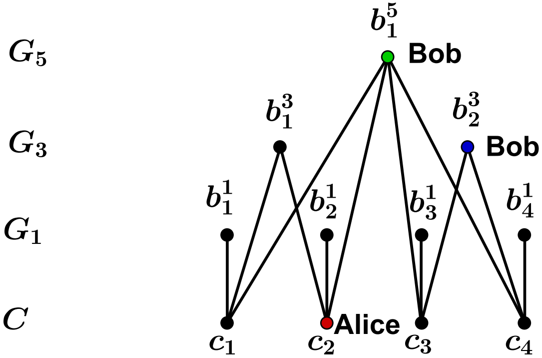
G_3
a_2^3
a_1^3
a_3^3
a_1^1
G_1
a_2^4
a_1^4
a_3^4
G_4
B_1^1
b_1^1
c_1
c_2
c_{2^k+1}
c_{2^k}
a_8^6
a_4^6
a_1^6
a_2^6
a_5^6
a_1^6
G_6
a_6^6
a_3^6
a_{15}^6
a_7^6
a_{15}^5
a_7^5
a_3^5
a_6^5
a_1^5
G_5
a_5^5
a_2^5
a_4^5
a_8^5

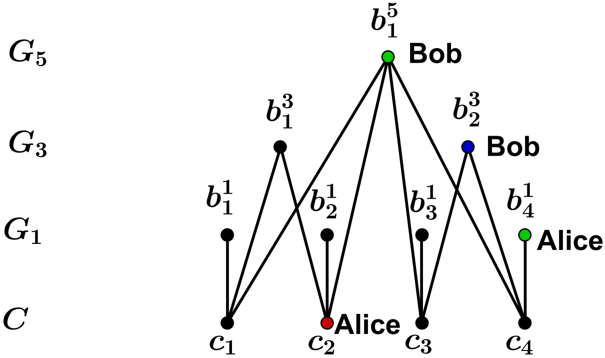

G_1
G_3
G_5
b_1^1
b_2^1
b_3^1
b_4^1
b_1^3
b_2^3
b_1^5
c_1
c_2
c_3
c_4

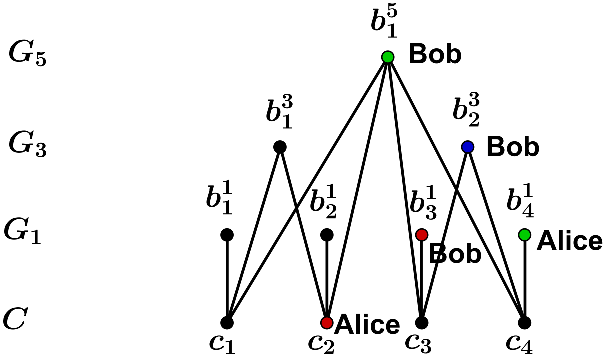

G_5
G_3
G_1
C
b_1^5
b_1^3
b_2^3
b_1^1
b_2^1
b_3^1
b_4^1
c_1
c_2
c_3
c_4


G_5
G_3
G_1
C
b_1^5
Bob
b_1^3
b_2^3
b_1^1
b_2^1
b_3^1
b_4^1
c_1
c_2
c_3
c_4


G_5
G_3
G_1
C
b_1^5
Bob
b_1^3
b_2^3
b_1^1
b_2^1
b_3^1
b_4^1
Alice
c_1
c_2
c_3
c_4


G_5
G_3
G_1
C
b_1^5
Bob
b_1^3
b_2^3
Bob
b_1^1
b_2^1
b_3^1
b_4^1
Alice
c_1
c_2
c_3
c_4


G_5
G_3
G_1
C
b_1^5
Bob
b_1^3
b_2^3
Bob
b_1^1
b_2^1
b_3^1
b_4^1
Alice
Alice
c_1
c_2
c_3
c_4


G_5
G_3
G_1
C
b_1^5
Bob
b_1^3
b_2^3
Bob
b_1^1
b_2^1
b_3^1
b_4^1
Alice
Bob
Alice
c_1
c_2
c_3
c_4

Definition 5

A *coloring number* of a graph G is the least integer k such that there exists a linear ordering of the vertices of G $v_1, \ldots, v_n$ in which every vertex v_i has at most $k-1$ neighbors v_j, where $j < i$. We denote it by $\mathrm{col}(G)$.

Definition 5

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Game variant (Marking Game)
Alice and Bob are picking vertices of a graph G alternately constructing a linear order of the vertices. The goal of Alice is to minimize the number of backward neighbors of every vertex, while Bob tries to maximize it.
The least number k guaranteeing that Alice has a strategy for keeping the number of backward neighbors of each vertex strictly below k is the *game coloring number* of a graph G, denoted by $\mathrm{col}_g(G)$.

Observation 6

Every graph G satisfies $\mu_g(G) \leq \mathrm{col}_g(G)$.

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Corollary 7

For any tree T, $\mu_g(T) \leq 4$.
For any outerplanar graph G, $\mu_g(G) \leq 7$.
For any planar graph G, $\mu_g(G) \leq 17$.

Parameter $\mu_g(G)$ is bounded for graphs with $\mathrm{col}(G) = 2$.
Is that true for graphs with $\mathrm{col}(G) = 3$?

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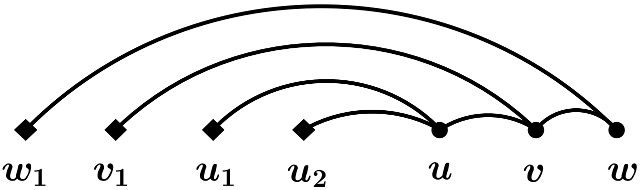
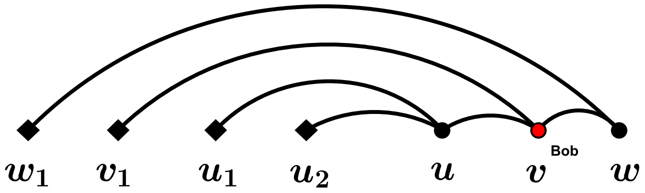
Theorem 8

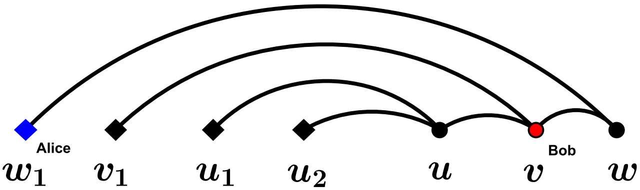
For every positive integer k there exists a graph G_k with $\mathrm{col}(G_k) = 3$ and $\mu_g(G_k) > k$.



Graph S



Graph S



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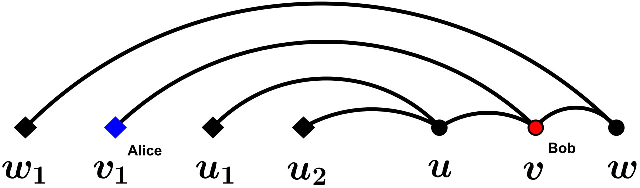
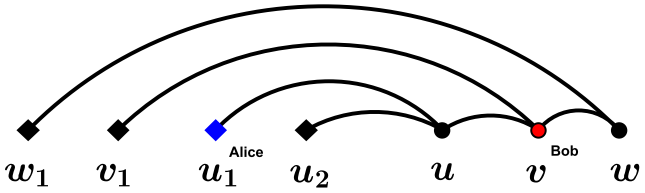


Graph S



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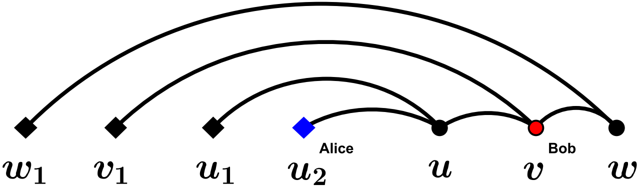

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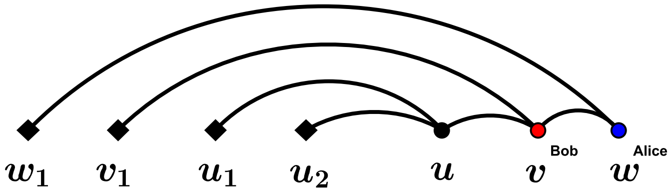
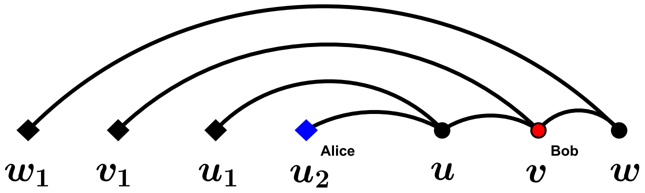
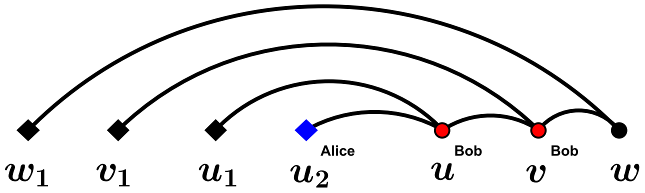


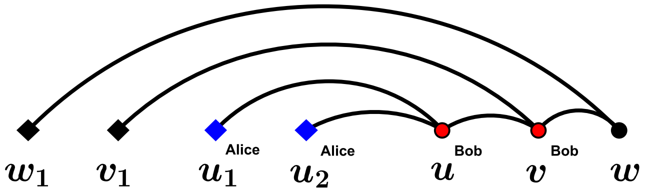
Graph S



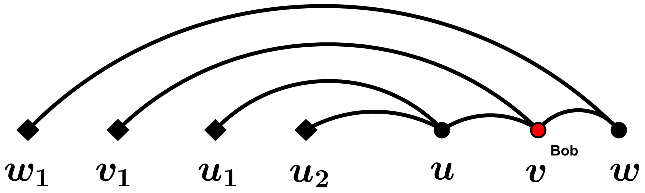
Graph S



Graph S



Graph S



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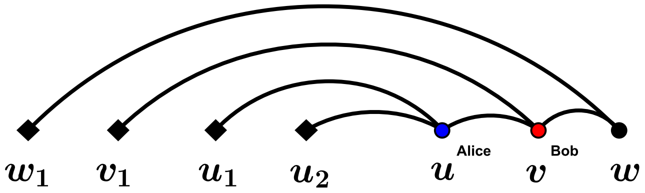

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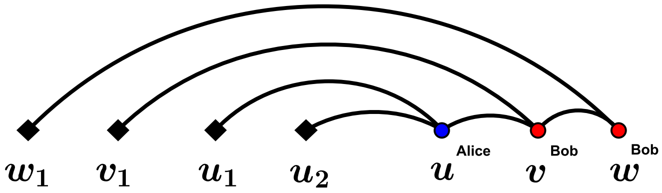

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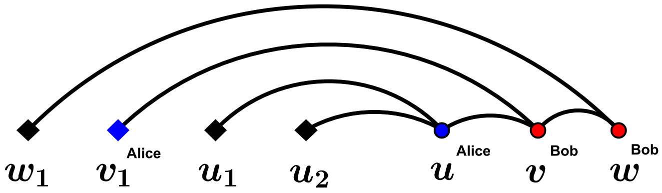
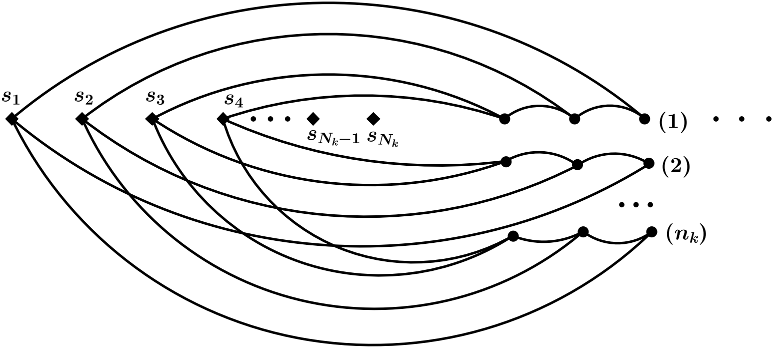


Graph S


s_1
s_2
s_3
s_4
s_{N_k-1}
s_{N_k}
(1)
(2)
(n_k)

THANK YOU FOR YOUR ATTENTION!!!