On upper bounds for the order of cages

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joint work with **Robert Jajcay**

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Origins

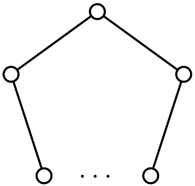
- first studied by William T. Tutte in 1947
- Ferenc Kárteszi studied a related problem in 1960 with Hamiltonian graphs
- around the same time Moore graphs were introduced and studied by Alan J. Hoffman and Robert R. Singleton in 1960; they were named after Edward F. Moore
- closely related to diameter–degree problem
  - graph with diameter d has girth at most $2d + 1$
  - bipartite graph with diameter d has girth at most $2d$

Definition: (k, g)–**cage**

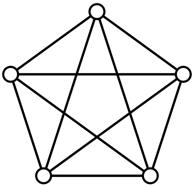
Let $k, g \in \mathbb{N}$, $k \geq 2$, $g \geq 3$.

Then a (k, g)-**graph** is a k-regular graph with girth g (simple and undirected).

And a (k, g)-**cage** is a (k, g)-graph with the least possible number of vertices.



$C_n \equiv (2, n)$-cage



$K_n \equiv (n-1, 3)$-cage

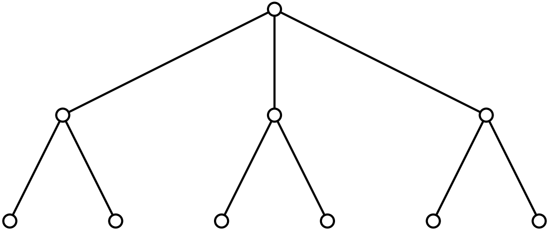
Definition: Moore bound

Moore bound

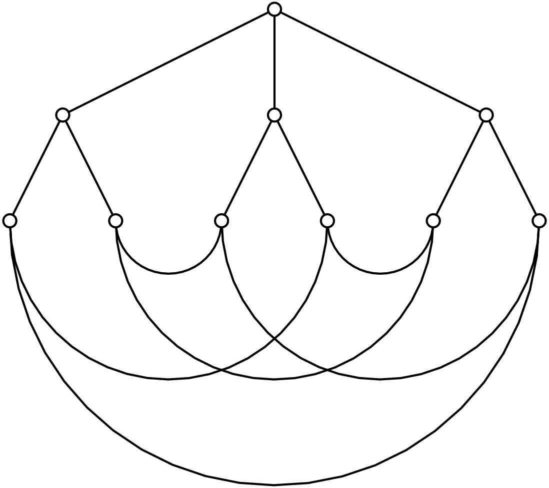
$$M(k,g) = \begin{cases} \frac{k(k-1)^{(g-1)/2}-2}{k-2}, & g \text{ odd}, \\ \frac{2(k-1)^{g/2}-2}{k-2}, & g \text{ even}. \end{cases}$$

- number of vertices needed in a (k,g)-graph
- obvious lower bound for a (k,g)-cage

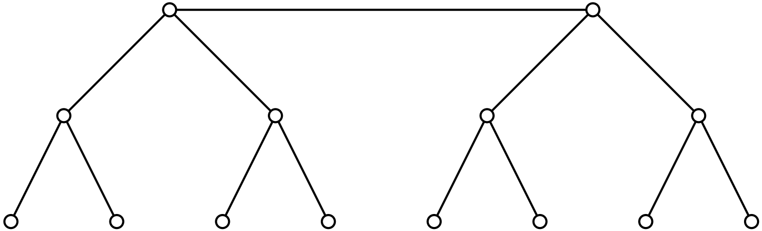
Example: $k = 3$, $g = 5$



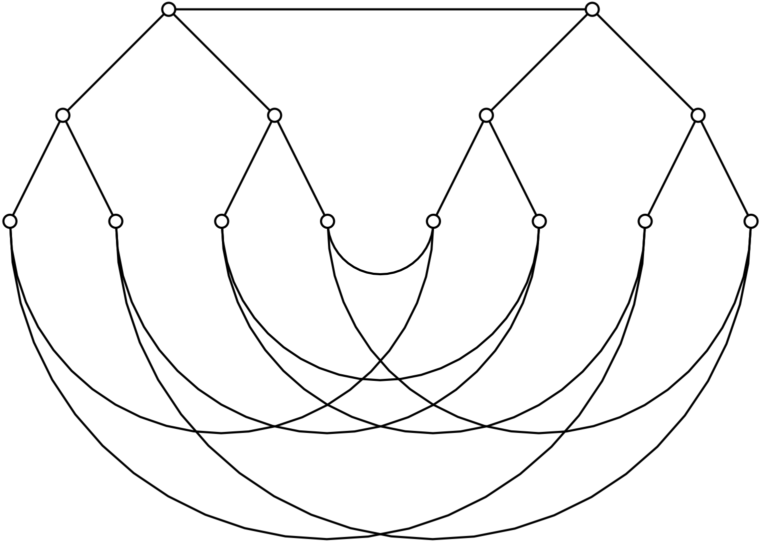
Example: Petersen graph



Example: $k = 3$, $g = 6$



Example: Heawood graph



Definition: Moore graph

A (k, g)-graph with $M(k, g)$ many vertices is called a **Moore graph**.

- a Moore (k, g)-graph is clearly a (k, g)-cage

Theorem [Bannai, Ito (1981) & Damerell (2010)]

There exists a Moore graph of degree k and girth g if and only if

(i) $k = 2$ and $g \geq 3$; (cycles)

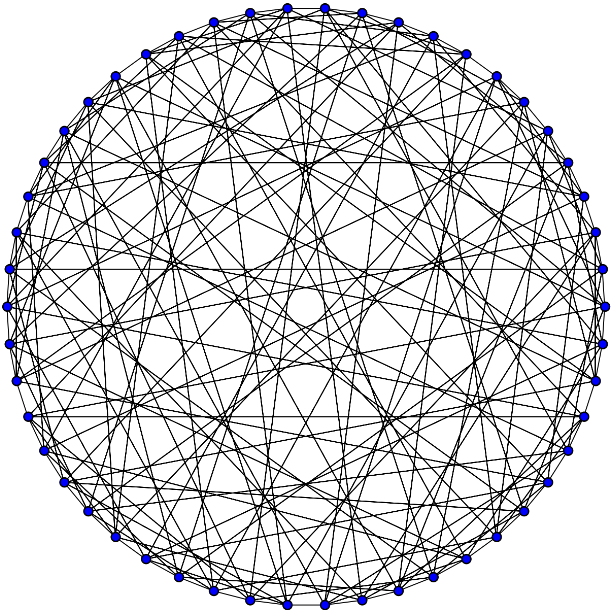
(ii) $k \geq 2$ and $g = 3$; (complete graphs)

(iii) $k \geq 2$ and $g = 4$; (complete bipartite graphs)

(iv) $g = 5$ and $k = 2$ (the 5-cycle), $k = 3$ (Petersen graph), $k = 7$ (Hoffman-Singleton graph), and possibly $k = 57$;

(v) $g = 6, 8$, or 12, if there exists a symmetric generalized polygon of order $k - 1$.

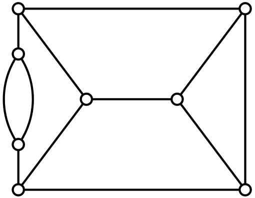
Definition: Moore graph



Definition: $n(k,g)$ **and** $rec(k,g)$

The number of vertices of a (k,g)-cage is denoted by $n(k,g)$.

The number of vertices of a (k,g)-graph which is currently the smallest known (k,g)-graph is denoted by $rec(k,g)$. (*the current record holder*)



base graph that gives voltage

$(3,14)$-graph with 384 vertices

$rec(3,14) = 384$

$$M(k,g) \leq n(k,g) \leq rec(k,g)$$

Existence of cages

– first proof of existence: Sachs, 1963

$\hookrightarrow$ first and **only** recursive constructive proof of (k, g)-graphs.

Theorem [Erdös, Sachs, 1963]

For every $k \geq 2$, $g \geq 3$,

$$n(k, g) \leq 4 \sum_{t=1}^{g-2} (k-1)^t .$$

Upper bounds I

Theorem [Erdös, Sachs, 1963]

For every $k \geq 3$, and every **odd** $g \geq 3$,

$$n(k, g+1) \leq 2n(k, g)\,.$$

Theorem [Balbuena, González-Moreno, Montellano-Ballesteros, 2013]

For every $k \geq 2$, and every **odd** $g \geq 5$,

$$n(k, g+1) \leq \begin{cases} 2n(k,g) - 2\frac{k(k-1)^{(g-3)/4}-2}{k-2}\,, & g \equiv 3 \pmod 4 \\ 2n(k,g) - 4\frac{(k-1)^{(g-1)/4}-1}{k-2}\,, & \text{otherwise.} \end{cases}$$

Upper bounds II

Theorem [Sauer, 1967]

For every $k \geq 2$ and $g \geq 3$,

$$n(k,g) \leq \begin{cases} 2(k-2)^{g-2}, & g \text{ odd}, \\ 4(k-1)^{g-3}, & g \text{ even}. \end{cases}$$

Sauer bound

Theorem [Sauer, 1967]

For every $g \geq 3$,

$$n(3,g) \leq \begin{cases} \frac{29}{12}2^{g-2} + \frac{2}{3}, & g \text{ odd}, \\ \frac{29}{12}2^{g-2} + \frac{4}{3}, & g \text{ even}. \end{cases}$$

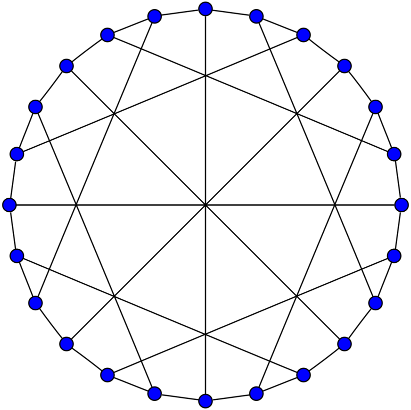
Upper bounds
– comparison for $k = 3$

g	Moore bound	Erdös, Sachs	Sauer
4	6	24	11
5	10	56	20
6	14	120	40
7	22	248	78
8	30	504	156
9	46	1016	310
10	62	2040	620
11	94	4088	1238
12	126	8184	2476

Current record holders

$k \backslash g$	5	6	7	8	9	10	11	12
3	**10**	**14**	**24**	**30**	**58**	**70**	**112**	**126**
4	**19**	**26**	**67**	80	275	384		728
5	**30**	**42**	152	170		1296	2688	2730
6	**40**	**62**	294	312				7812
7	**50**	**90**		672				32928
8	80	**114**		800				39216
9	96	**146**	1152	1170			74752	74898
10	124	**182**		1640				132860

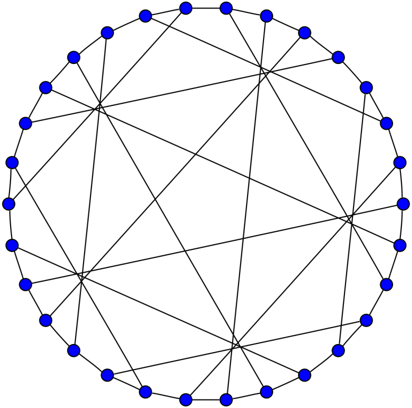
Knowns cages: McGee graph



- first 3-valent cage which is not a Moore graph:

$$n(3,7) = 24, M(3,7) = 22$$

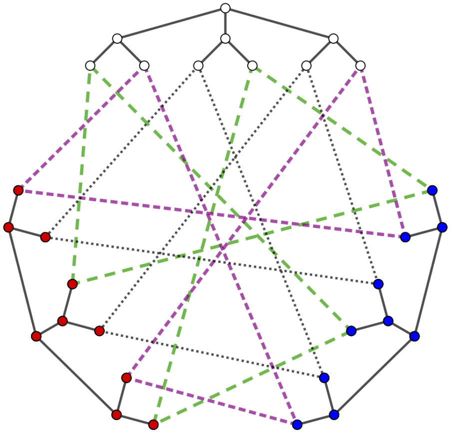
Knowns cages: Tutte's cage



- a Moore graph:

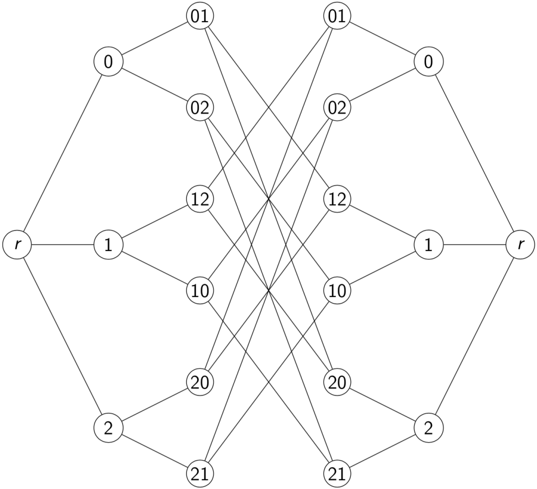
$$n(3,8) = M(3,8) = 30$$

My construction I



$$n(k, g) \leq 3 \cdot M(k, g-1)$$

My construction II



$$n(k, g) \leq 2 \cdot M(k, g - 1)$$

Work in progress

- various individual constructions for $k = 3$ and $g \leq 10$
  $\hookrightarrow$ still working on a generalization of some to an arbitrary $g \geq 5$

- various individual constructions for $k = 4, 5$ and $g \leq 8$
  $\hookrightarrow$ still working on a generalization of some to an arbitrary $k \geq 4$ (and arbitrary $g \geq 5$)

Conjecture I.

$$n(k, g) \leq 2 \cdot M(k, g - 1)$$

Conjecture II.

$$n(k, g) \leq k \cdot M(k, g - 1)$$

THANK YOU! :)