Hamiltonicity of cubic Cayley graphs of small girth

dedicated to Brian's 80-th and Dragan's 65

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Cubic graphs of large cyclic connectivity

- Conjecture T (C. Thomassen): If the cyclic connectivity of a cubic graph X is large, then X is hamiltonian.
- Conjecture T* (the strongest version of A): **Every 7-cyclically connected cubic graph except the Coxeter graph is hamiltonian.**
- Remark: 7 in Conj. T^* cannot be replaced by 6, because there are infine families of cyclically 6-connected snarks, in fact they form an NP-class of cubic graphs.
- Conjectures T and T* are very strong, in particular, a positive solution of T* would imply that there no cyclically 7-connected snarks, thus confirming in the affirmative the **Jaeger's conjecture** open since 1979.

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Hamiltonicity in cubic Cayley graphs

A folklore conjecture inspired by the Lovasz conjecture:

Conjecture F: **Every Cayley graph is hamiltonian.**

Assume T* holds, then to prove Conj. F for cubic Cayley graphs we have to deal with the following problem:

Problem: **Prove that cubic Cayley graphs of girth at most six are hamiltonian.**

Note that N. and Škoviera proved in 1995 that for a cubic vertex-transitive graph the cyclic connectivity is equal to the girth!

Hence, the cyclic connectivity c, implies there exists a cycle of length c, and this implies that there exists a relation of length c in terms of the generators.

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Type I. $X = Cay(G; a, b, c)$, where $a^2 = b^2 = c^2 = 1$

We may assume that $|ab| \leq |ac| \leq |bc|$, G is a finite quotient of the extended triangle group of type (k, m, n), $k \leq m \leq n$.

Theorem

If the girth $g(X) \leq 6$, then one of the following happens:

- $g(X) = 3$, $G = C_2 \times C_2$ *and* $X \cong K_4$,
- $g(X) = 4$, *and* $(ab)^2 = 1$,
- $g(X) = 6$, *and* $(abc)^2 = 1$, *and* X *is a honeycomb graph,*
- $g(X) = 6$, *and* $(ab)^3 = 1$. *(the difficult case)*

Hamiltonicity of graphs of Type I, case $(ab)^2 = 1$

Proposition: Graphs of type I satisfying $(ab)^2 = 1$ are hamiltonian.

- Proved by Rappaport-Strasser, see Pak, Radoičič DM 2009 for the proof,
- It follows from a result by Powers (1985), who proved that Cayley cubic graphs of girth 4 are hamiltonian,
- There is a proof based on the method by **Glower and Marušič**

Essence of G-M method

Let X be a cubic strongly embedded graph into a surface (the faces are bounded by true cycles). Let the of faces can be 3-coloured $F = F_1 \cup F_2 \cup F_3$ such that

- F_1 is independent, and the collection of cycles bounding faces in F_1 forms a 2-factor,
- F_2 is independent,
- F_3 induces a tree in the dual.

Then X admits a contractible hamilton cycle.

Case I $(ab)^2 = 1$: proof by G-M. method

- form an embedding of X into a surface by attaching a 2-cell to all the (ab)-cycles, (bc)-cycles and (ac)-cycles.
- consider the partial dual $Y = X^*$, induced by the vertices that correspond to the (ab)-cycles and (bc)-cycles.
- observation Y is a bipartite graph, where all the (ab)-vertices are of degree two,
- take a spanning tree T of Y and form a vertex decomposition into an induced tree T' and an independent set I by setting I to be the set of (ab)-vertices that are of degree 1 in T.
- by G-M. T' determines in the embedding of X a tree of faces bounded by a (contractible) hamilton cycle.

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Case II. $(abc)^2 = 1$

Proposition: Graphs of type I $(abc)^2 = 1$ are hamiltonian.

Outline of the proof:

- Observe that each edge of X lies in exactly two 6-cycles induced by the relation $(abc)^2 = 1$,
- Thus X is a honeycomb graph on the torus,
- Honeycomb graphs are hamiltonian, see B. Alspach and D. Matthew (2009), or Yang et.all (2004)

Case I $(ab)^3 = 1$, the difficult case

Wanted: Cayley graphs coming from finite (torsion free) quotients of the extedned triangle group:

$\Delta(3,m,n) = \langle a,b,c|\ a^2 = b^2 = c^2 = (ab)^3 = (ac)^m = (bc)^n = 1\rangle$

are hamiltonian.

G-M. method gives almost the result for $(k,m,n) = (3,3,n)$!

Graphs of type II.

Theorem

Let X be a Cayley cubic graph $X = Cay(G; a, b)$, $a^2 = 1$, of girth $g(X) \leq 6$. Then X one of the following cases happens:

- $g(X) = 3$, *and* $b^3 = 1$, *or* $G \cong C_4$ *and* $X \cong K_4$,
- $g(X) = 4$, $a = b^3$, $G \cong C_6$ *and* X *is* $K_{3,3}$,
- $g(X) = 4$, $aba = b^{\pm 1}$, G *is abelian, or dihedral, and* X *is, a prism or a Mobius ladder,*
- $g(X) = 4$, *and* $b^4 = 1$,
- $g(X) = 5$, *and* $b^5 = 1$,
- $g(X) = 6$, *and* $G = \langle a, b \mid a^2 = b^8 = 1, aba = b^{\pm 3} \rangle$, X *is the generalised Petersen graph* $GP(8, 3)$,
- $g(X) = 6$, $ab^2a = b^{\pm 2}$ *and* X *is a honeycomb graph,*
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- $g(X) = 6$, *and either* $(ab)^3 = 1$, *or* $b^6 = 1$.

Hamiltonicity, the difficult cases

The difficult cases are:

- $b^3 = 1$, this case can be solved by using Conjecture T*
- $b^5 = 1$, $b^6 = 1$, no idea how to solve these cases,
- $(ab)^3 = 1$, **G.M. method** gives existence of hamilton path, and in most cases a hamilton cycle as well,

In all the other cases we can verify the hamiltonicity.