# *The Hamilton-Waterloo Problem with Cycle Sizes of Different Parity*

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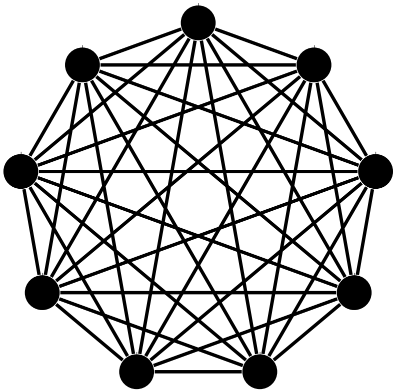
Graphs, groups and more
May 31st, 2018

# $K_9$ *into* 4 $C_3$*-factors*



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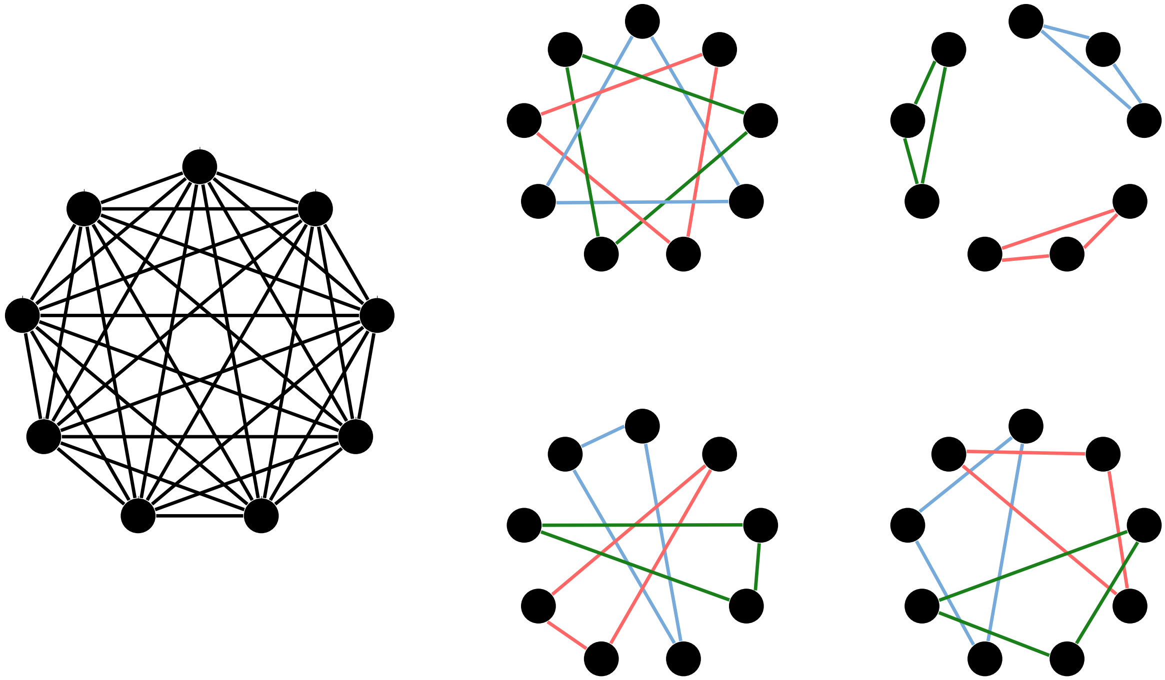
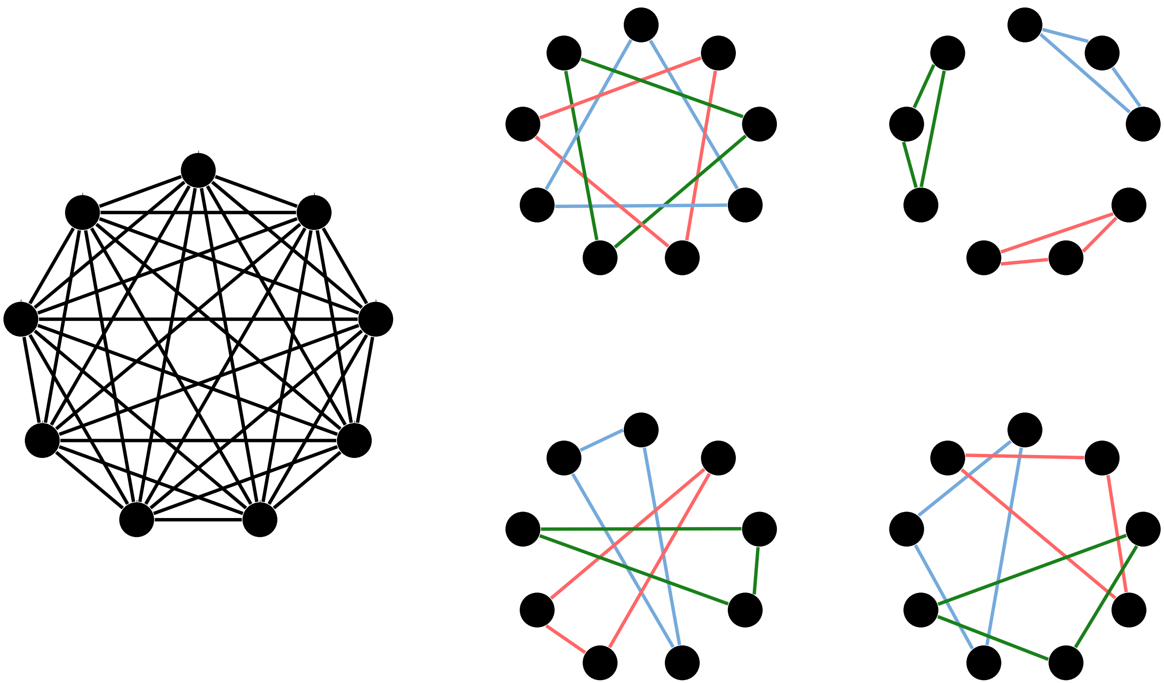
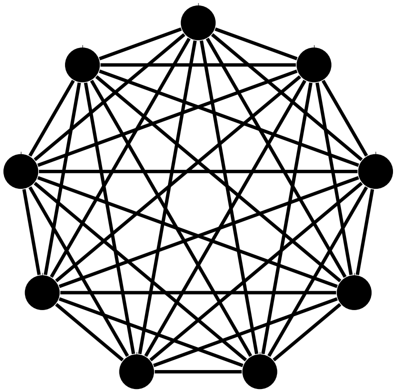
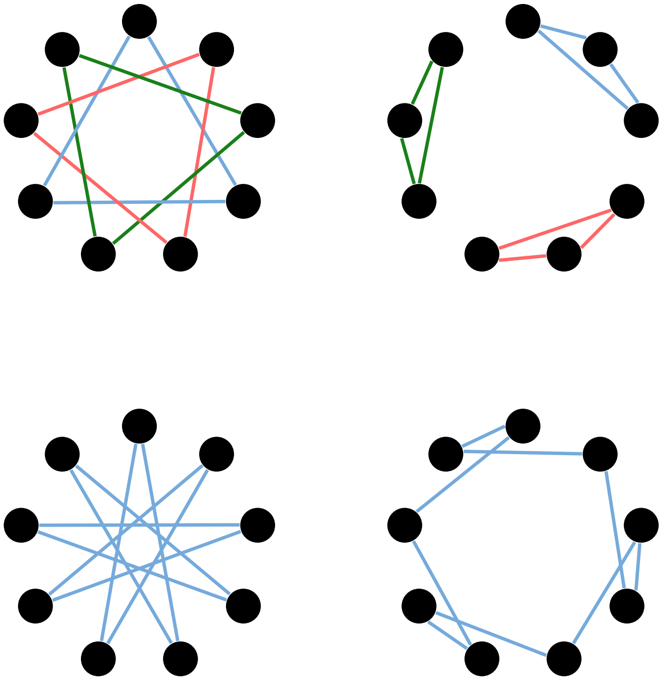


# $K_9$ *into* 4 $C_3$-*factors and* 0 $C_9$-*factors*



# $K_9$ into 2 $C_3$-factors and 2 $C_9$-factors





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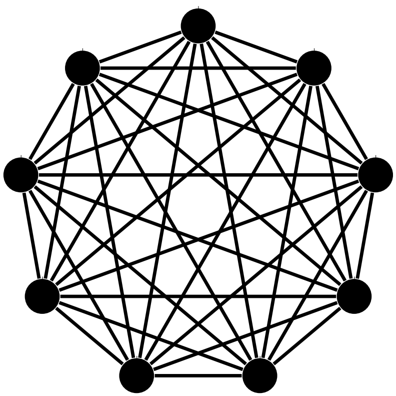

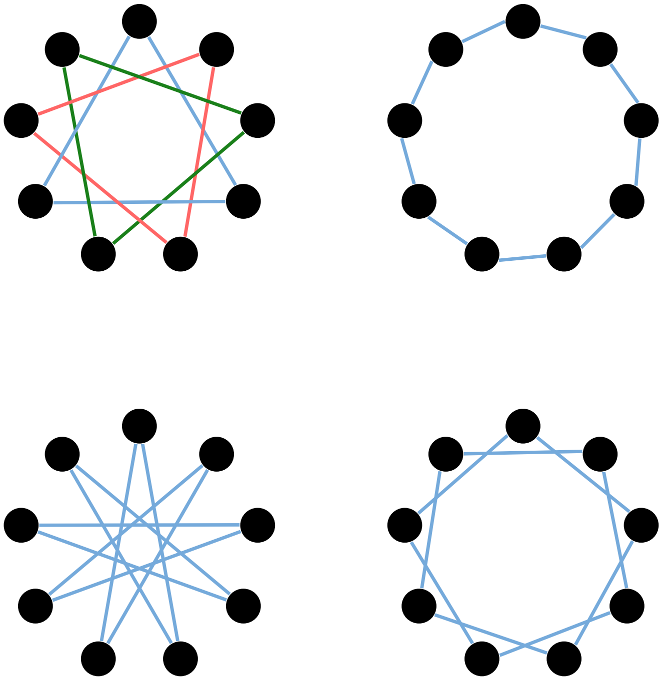

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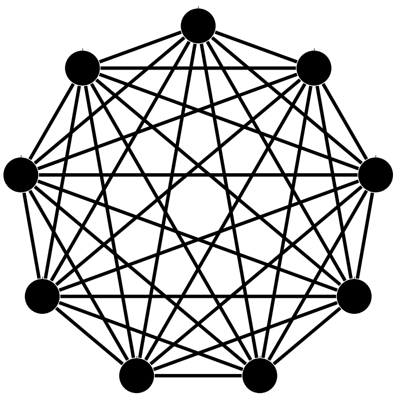
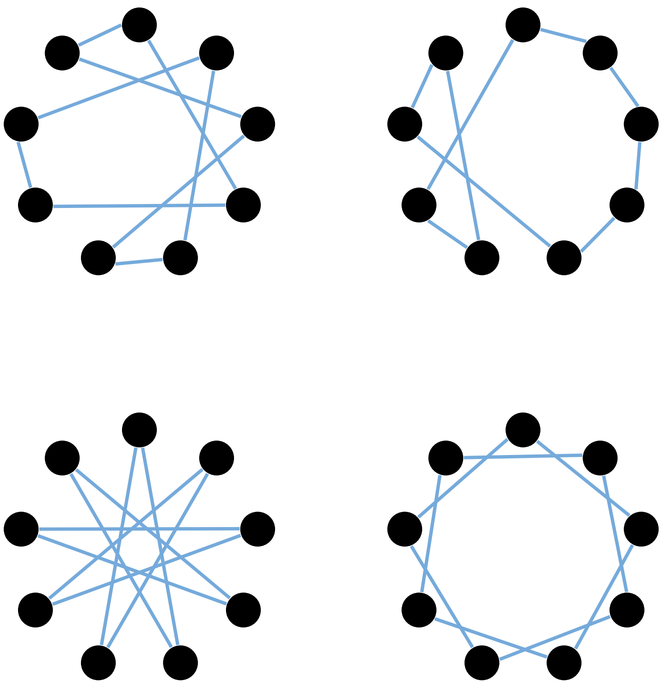
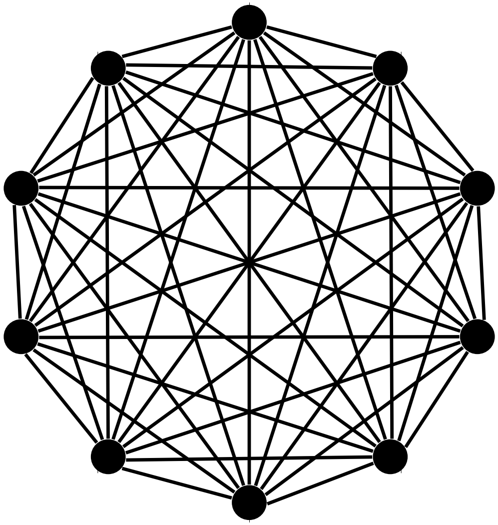




# $K_{10}$ *into a* 1-*factor,* 2 $C_{10}$-*factors, and* 2 $C_5$-*factors*



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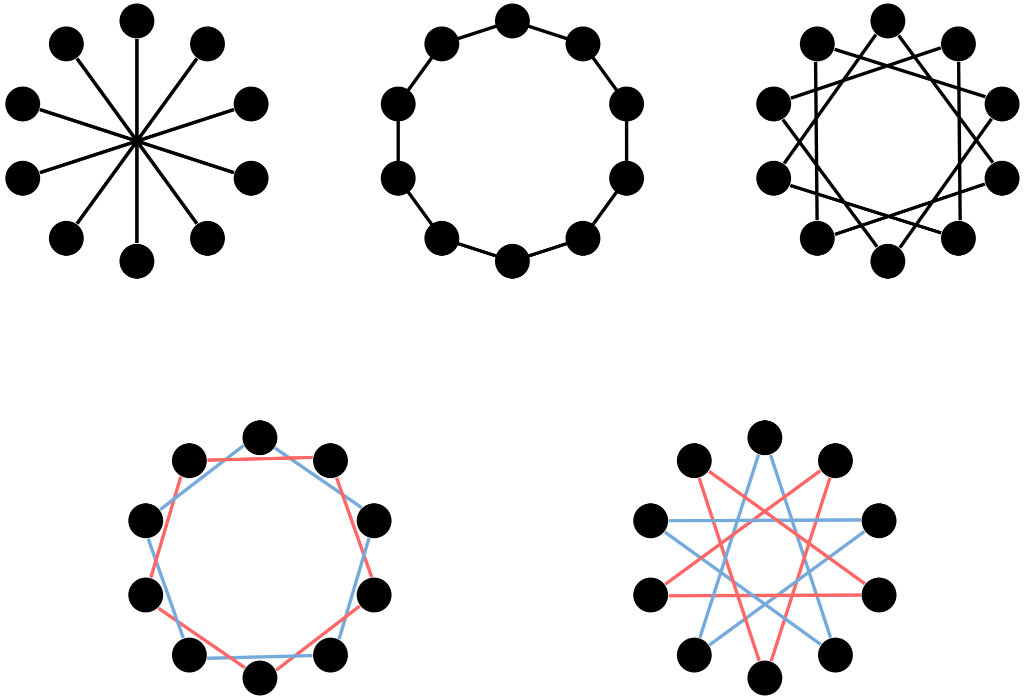

# The Uniform Hamilton-Waterloo Problem for Complete Graphs

## Question

Given $m$, $v$ and $w$, such that $v$ and $w$ divide $m$, the Uniform Hamilton-Waterloo Problem asks whether $K_m$ can be decomposed into $r$ $C_v$-factors and $s$ $C_w$-factors (and a 1-factor) for every $r + s = (m-1)/2$ $(r + s = (m-2)/2)$.

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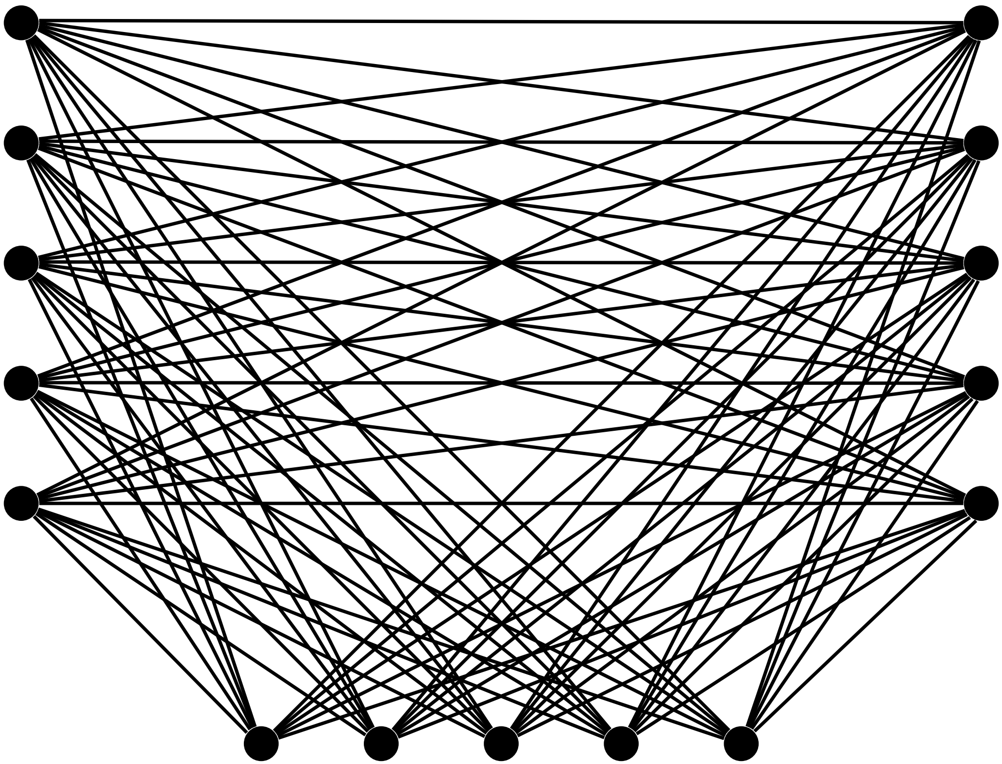
- Thus, we work on $v, w$ such that $\gcd(v, w) \geq 3$, with $w$ odd, and allowing $v$ to be either even or odd.

# $K_{(5:3)}$ *into 2 $C_3$-factors and 3 $C_{15}$-factors*



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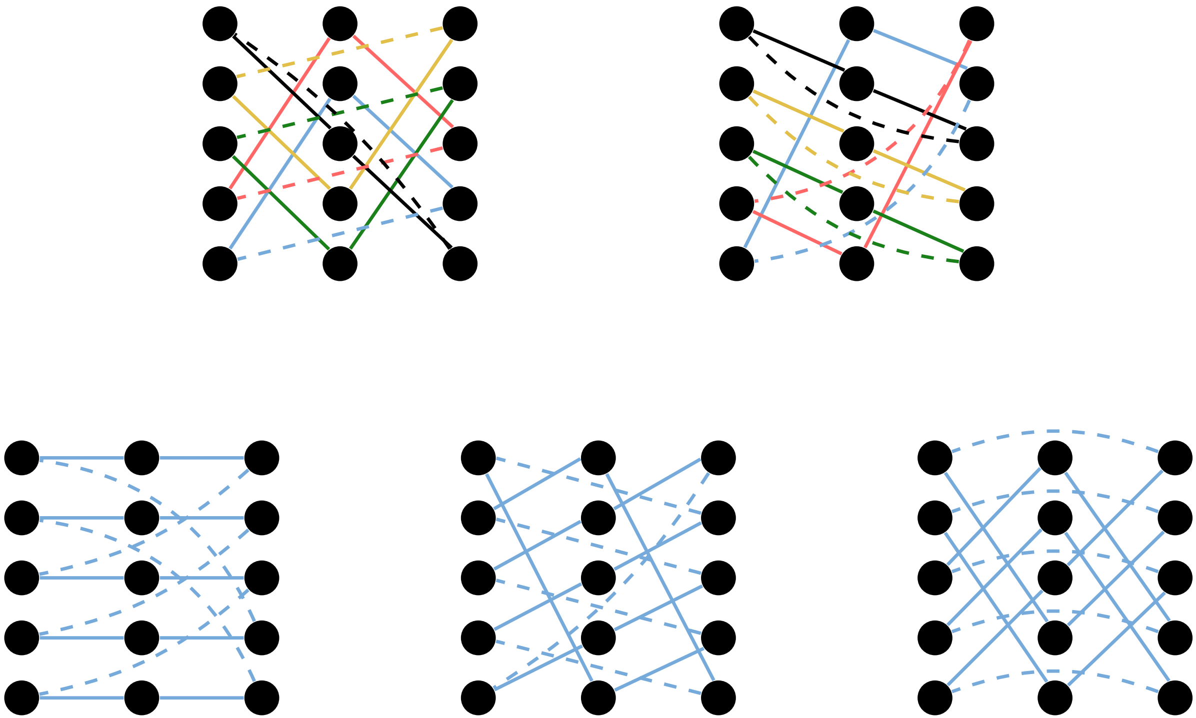

## Needed Known Results

*Theorem (Alspach and Haggkvist, 1985; Alspach, Schellenberg, Stinson and Wagner, 1989; Hoffman and Schellenberg 1991; Ray-Chadhuri and Wilson, 1971)*

*$K_m$ (or $K_m - F$ if $m$ is even) can be decomposed into $C_v$-factors if and only if $m \equiv 0 \pmod{v}$, $(m, v) \neq (6, 3)$ and $(m, v) \neq (12, 3)$.*

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**Theorem (Liu)**

*If $v \geq 3$ and $t \geq 2$, $K_{(m:t)}$ can be decomposed into $C_v$-factors if and only if $v$ divides $(mt)$, $m(t-1)$ is even, $c$ is even if $t = 2$, and $(m, t, c) \notin \{(2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)\}$.*

# Main Result

**Theorem (KP)**

*Let $m, k, v$, and $w$ be positive integers with $\frac{m\gcd(v,w)}{4^k vw} \geq 3$ an integer, $\gcd(v,w) \geq 3$, $v, w$ odd. Then $K_m$ can be decomposed into $s$ $C_{2^k v}$-factors, $r$ $C_w$-factors and a 1-factor for every $s, r \neq 1$.*

# *Construction schematics*

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- after giving weight we get $K_{(v_1w_1\gcd(v,w):t)}$ decomposed into the $C_{\gcd(v,w)}$-factors with weight,

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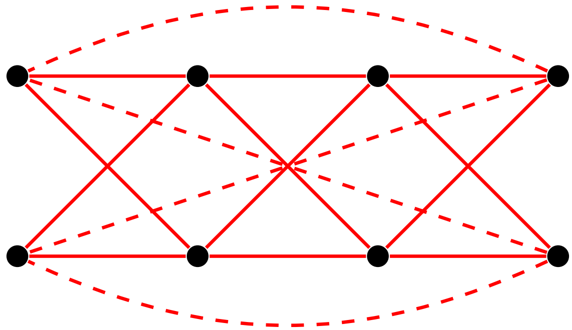
- give weight $v_1w_1$ to each vertex,
- after giving weight we get $K_{(v_1w_1\gcd(v,w):t)}$ decomposed into the $C_{\gcd(v,w)}$-factors with weight,
- the edges of $K_m$ not used give copies of $K_{\gcd(v,w)v_1w_1}$,
- decompose each copy of $K_{\gcd(v,w)v_1w_1}$ into $C_v$-factors or into $C_w$-factors, and find the necessary decompositions of the $C_{\gcd(v,w)}$-factors, after giving them weight.

# $C_{(2:4)}$

$G = C_{(2:4)}$



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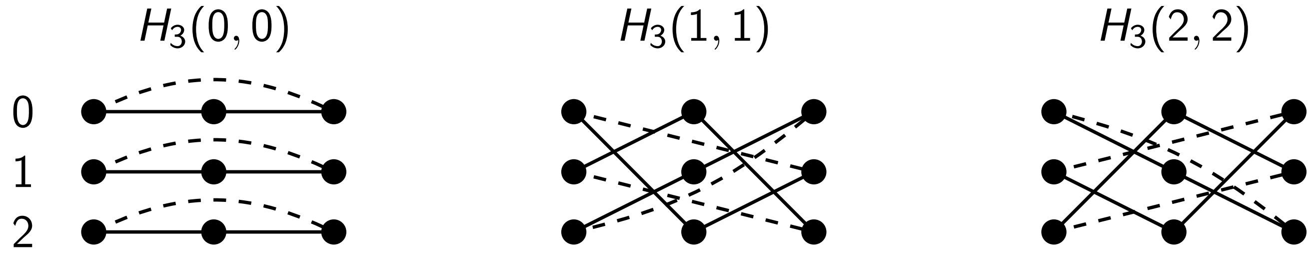

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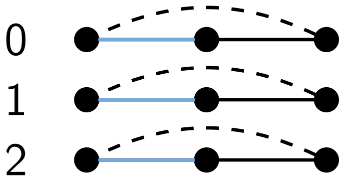
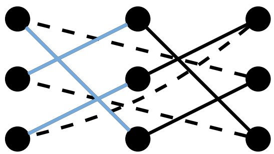
$H_3(0,0)$



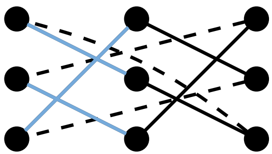

$+2 \cdot 0$

$H_3(1,1)$



$+2 \cdot 1$

$H_3(2,2)$



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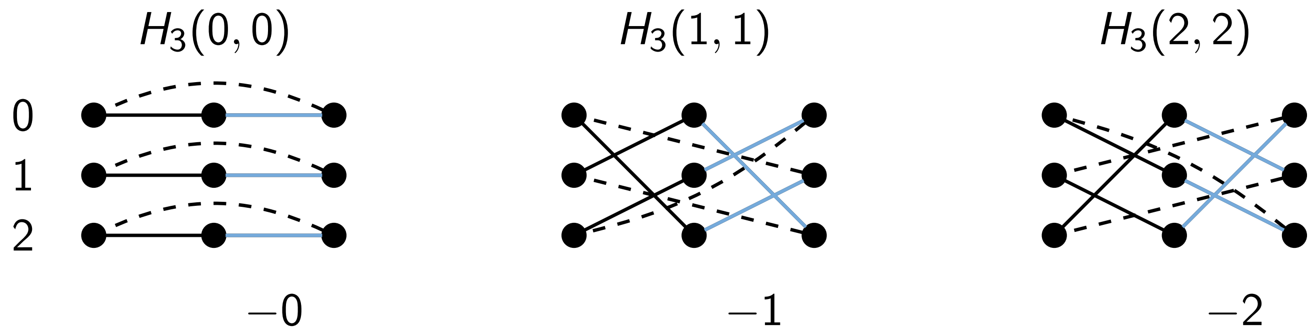

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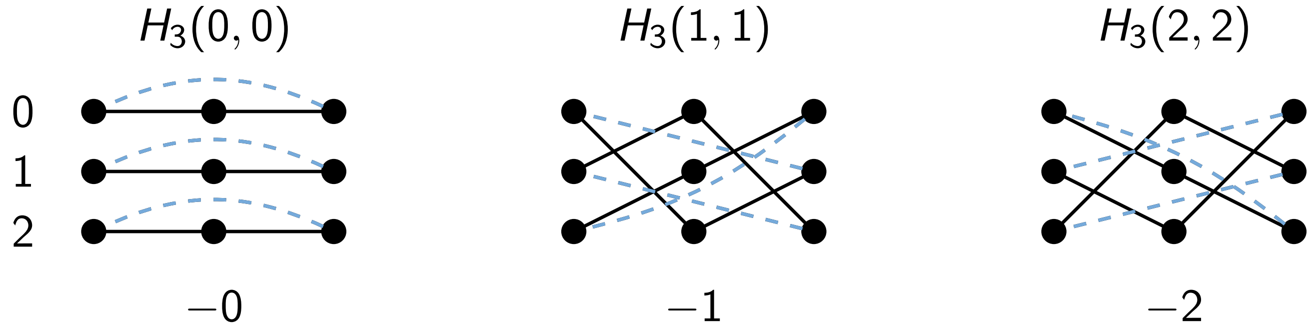

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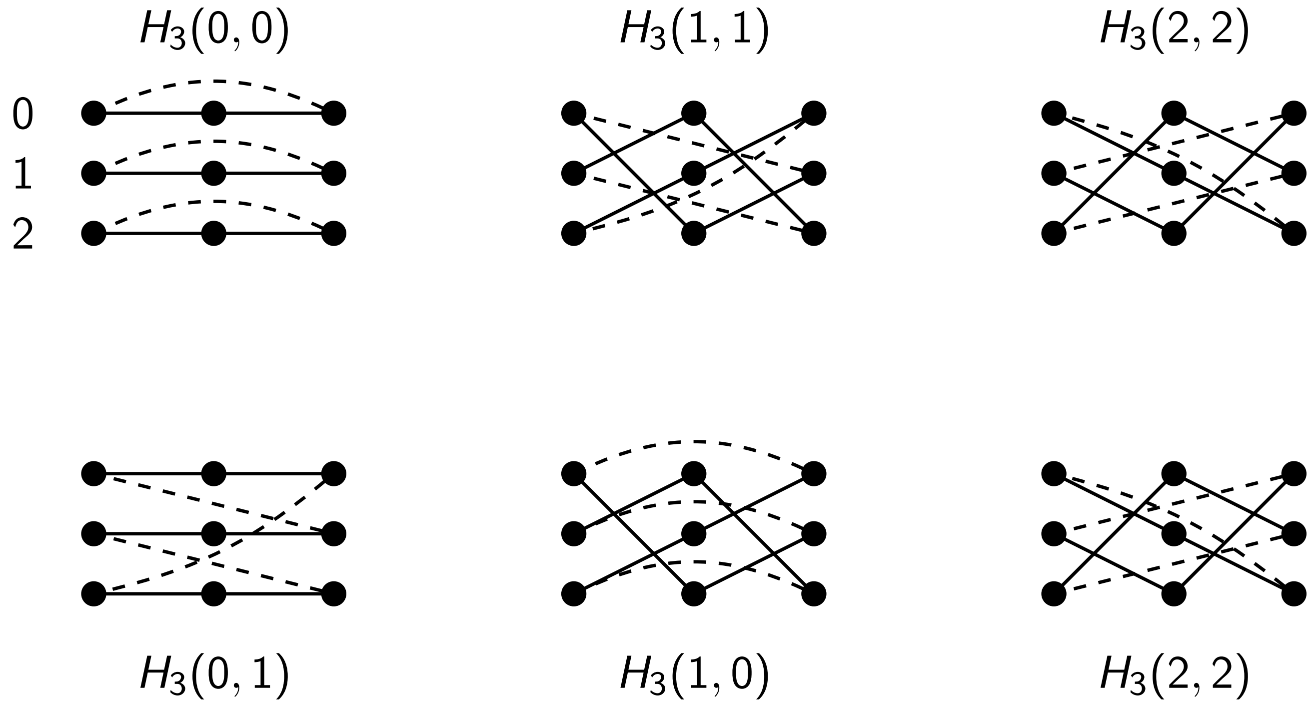

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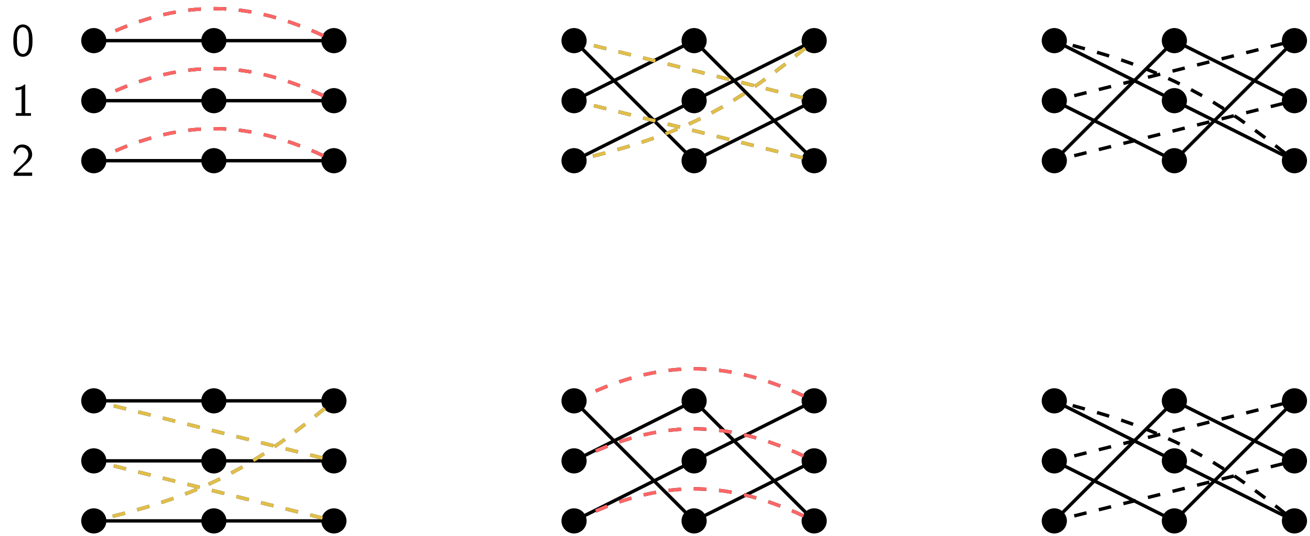
$H_3(0, 0)$ $H_3(1, 1)$ $H_3(2, 2)$




$H_3(0, 1)$ $H_3(1, 0)$ $H_3(2, 2)$

If $\phi$ is a permutation of $\{0, 1, ..., v_1 - 1\}$, then $C_{(v_1:t)}$ can be decomposed into $H_{v_1}(0, \phi(0)), \ldots, H_{v_1}(v_1 - 1, \phi(v_1 - 1))$.

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If $\gcd(i - \phi(i), v_1) = 1$, then $H_{v_1}(i, \phi(i))$ is a Hamilton cycle.

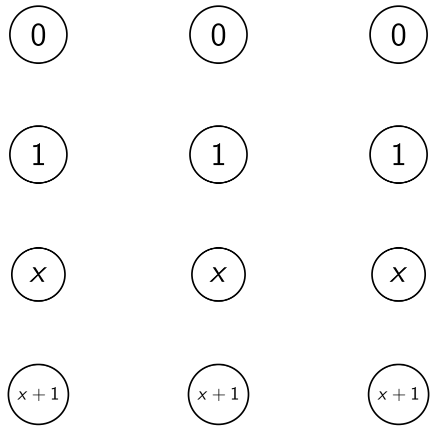
# Using Rings of Polynomials

We are going to decompose $C_{(4^k:t)}$ into $C_{2^k t}$-factors and $C_t$-factors.

# *Using Rings of Polynomials*

Consider the ring $R = \mathbb{Z}_{2^k}[x]/(x^2 + x + 1)$. Label the vertices of $C_{(4^k:t)}$ with the elements of $R$.

# Using Rings of Polynomials



For each $\alpha \in R$, define the bijection $f_\alpha(\gamma) = x\gamma + \alpha$. Note that $f_\beta \circ f_\alpha^2(\gamma) = \gamma - \alpha + \beta$, $f_\alpha^3(\gamma) = \gamma$.

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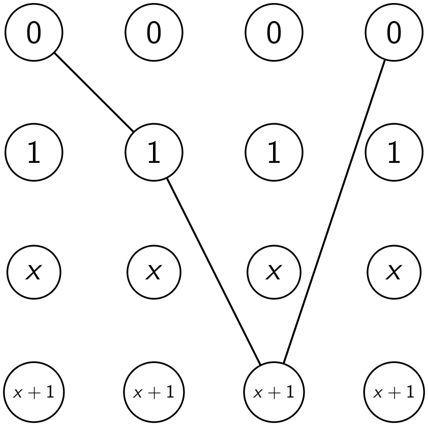
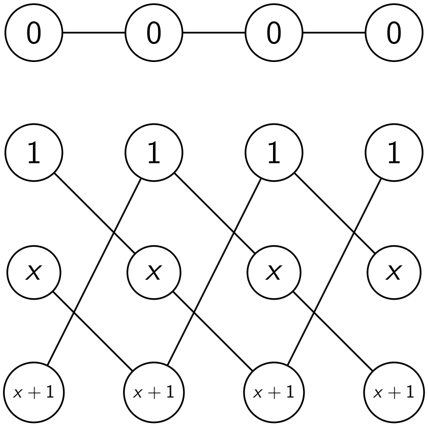
$f_1(\gamma) = x\gamma + 1$

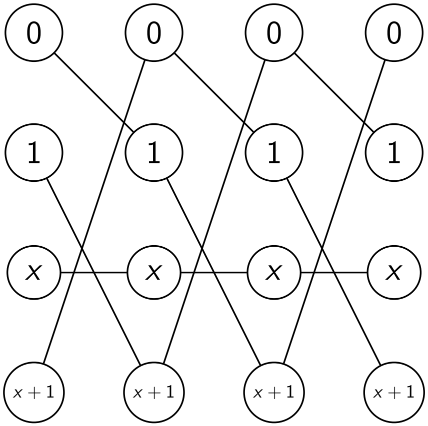


# Using Rings of Polynomials

$\Gamma(0,0)$

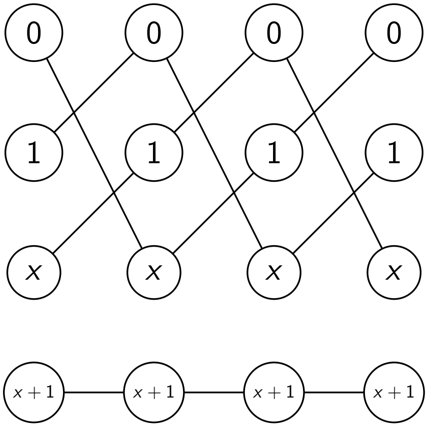



$\Gamma(1,1)$

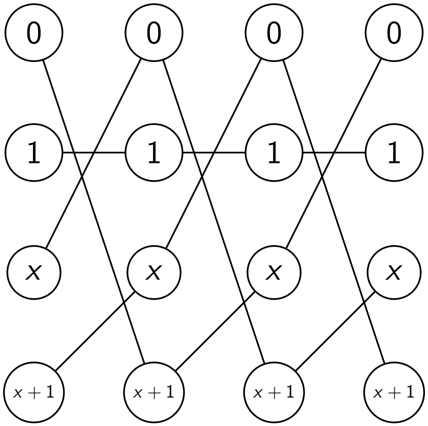


# *Using Rings of Polynomials*

$\Gamma(x, x)$




$\Gamma(x+1, x+1)$



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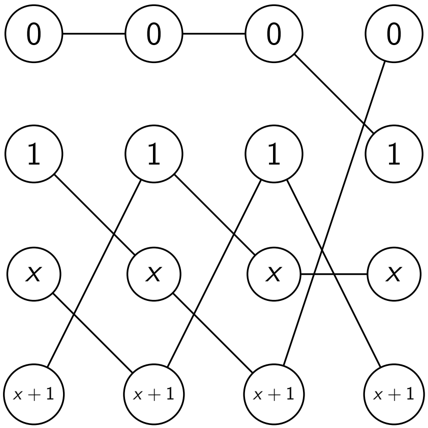


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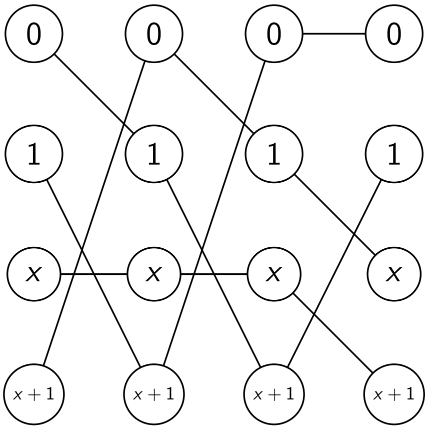

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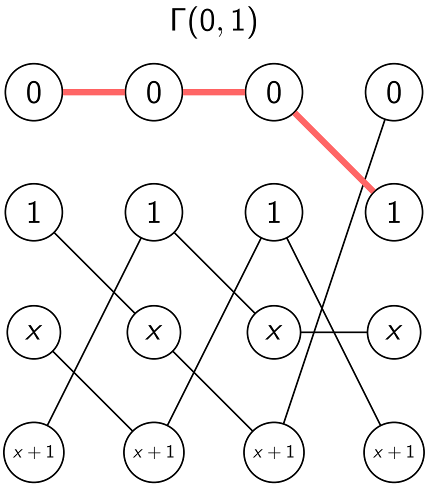


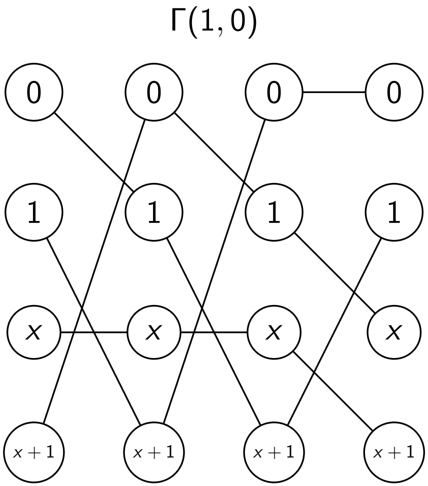

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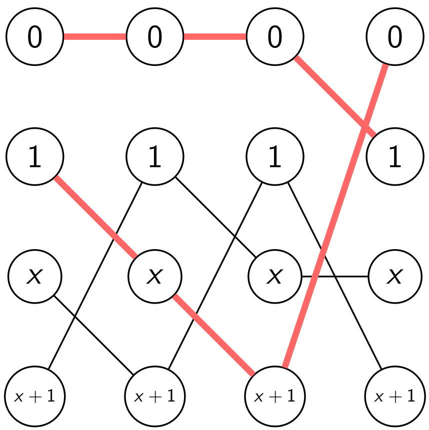


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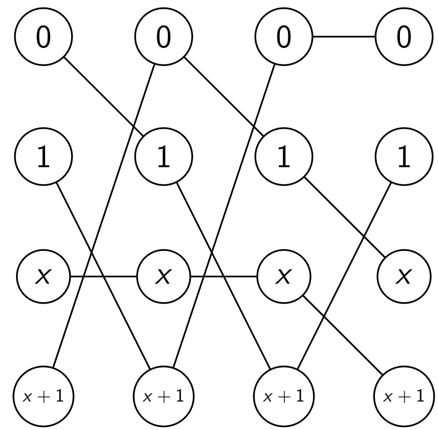

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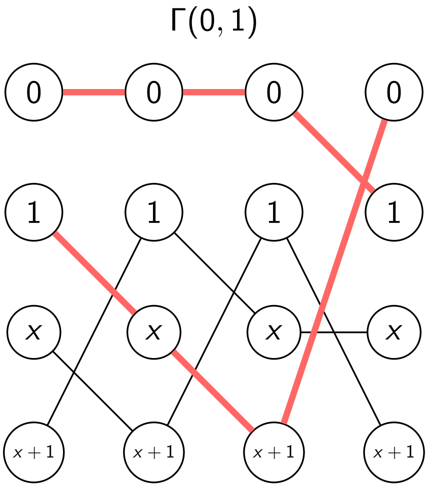


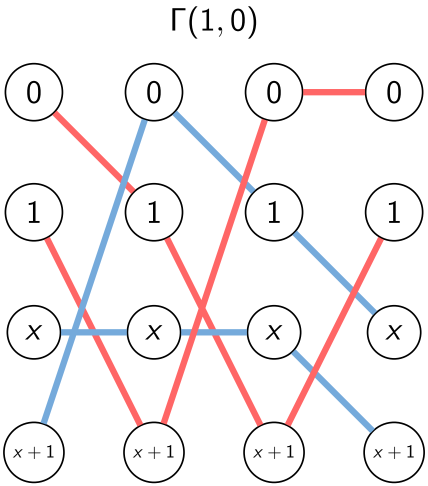

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- Decide whether decompositions with $r$ or $s$ equal to 1 can be achieved.

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- Decide whether decompositions with $r$ or $s$ equal to 1 can be achieved.
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- Find decompositions when $2^k$ divides $m$, but $4^k$ does not.
- Decide whether decompositions with $r$ or $s$ equal to 1 can be achieved.
- Study non-uniform decompositions.
- Study a different problem.

Thank you!!!