Hamilton decompositions of one-ended Cayley graphs

Florian Lehner

University of Warwick

with J. Erde and M. Pitz

Problem.

Does every Cayley graph of a finite Abelian group have a Hamilton decomposition? Alspach '84

Problem.

Does every Cayley graph of a finite Abelian group have a Hamilton decomposition? Alspach '84

Question.

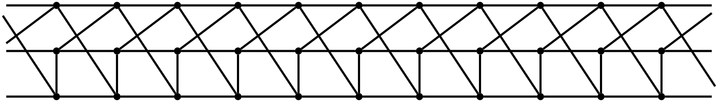
What about infinite groups?

Problem.
Does every Cayley graph of a finite Abelian group have a Hamilton decomposition? Alspach '84

Question.
What about infinite groups? Bryant, Herke, Maenhaut, Webb '17

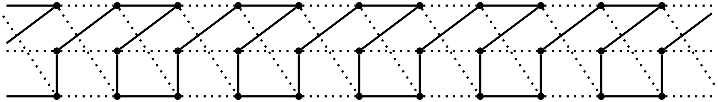
Theorem.
Any Cayley graph of a countable Abelian group has a Hamilton cycle.

Nash-Williams '59

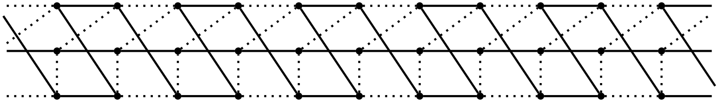


Infinite cycles and a necessary condition

Theorem.
Any Cayley graph of a countable Abelian group has a Hamilton cycle (double ray). Nash-Williams '59

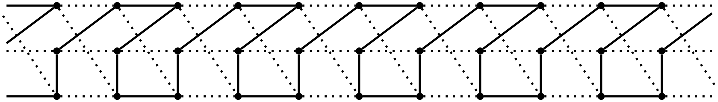


Theorem.
Any Cayley graph of a countable Abelian group has a Hamilton cycle (double ray). Nash-Williams '59



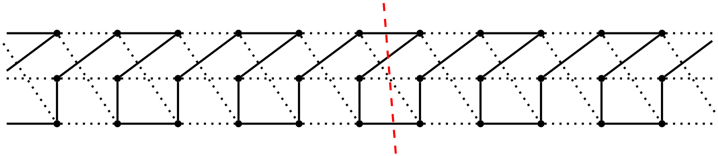
Infinite cycles and a necessary condition

Theorem.
Any Cayley graph of a countable Abelian group has a Hamilton cycle (double ray). Nash-Williams '59



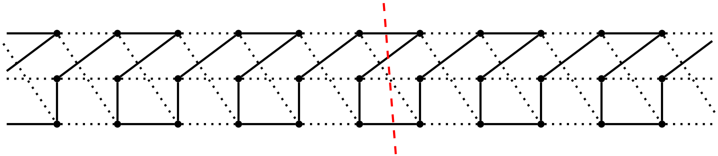
Infinite cycles and a necessary condition

Theorem.
Any Cayley graph of a countable Abelian group has a Hamilton cycle (double ray). Nash-Williams '59



Infinite cycles and a necessary condition

Theorem.
Any Cayley graph of a countable Abelian group has a Hamilton cycle (double ray). Nash-Williams '59



Observation.
Any $2k$-regular graph admitting a Hamilton decomposition satisfies

$(*)$ $|K| \equiv k \mod 2$ for every finite cut K with two infinite sides.

The 4-regular case

Theorem.
If a 4-regular Cayley graph of an Abelian group satisfies ⊛, then it has a Hamilton decomposition.

Finitely generated Abelian groups

Theorem. Let G be a finitely generated Abelian group. Then

$$G \simeq \mathbb{Z}^n \times F$$

for some $n \geq 0$ and some finite Abelian group F.

Finitely generated Abelian groups

Theorem. Let G be a finitely generated Abelian group. Then

$$G \simeq \mathbb{Z}^n \times F$$

for some $n \geq 0$ and some finite Abelian group F.

Definition. n is called the rank of G

- $n = 0$ finite groups
- $n = 1$ two-ended groups
- $n \geq 2$ one-ended groups

Theorem.
If a 4-regular Cayley graph of an Abelian group satisfies $\circledast$, then it has a Hamilton decomposition.

Theorem.
If a 4-regular Cayley graph of an Abelian group satisfies ⊛, then it has a Hamilton decomposition.

G finite Bermond, Favaron, Maheo '89

The 4-regular case

Theorem.
If a 4-regular Cayley graph of an Abelian group satisfies ⊛, then it has a Hamilton decomposition.

G finite	Bermond, Favaron, Maheo '89
two ends	$G = \mathbb{Z}$: Bryant, Herke, Maenhaut, Webb '17

The 4-regular case

Theorem.
If a 4-regular Cayley graph of an Abelian group satisfies ⊛, then it has a Hamilton decomposition.

G finite	Bermond, Favaron, Maheo '89
two ends	$G = \mathbb{Z}$: Bryant, Herke, Maenhaut, Webb '17
	G arbitrary: Erde, FL '18+

The 4-regular case

Theorem.
If a 4-regular Cayley graph of an Abelian group satisfies ⊛, then it has a Hamilton decomposition.

G finite	Bermond, Favaron, Maheo '89
two ends	$G = \mathbb{Z}$: Bryant, Herke, Maenhaut, Webb '17
	G arbitrary: Erde, FL '18+
one end	$G = \mathbb{Z}^2$

Main theorem

Theorem.
Let G be a one ended Abelian group, and let S be a generating set containing no torsion elements.
Then $\mathrm{Cay}(G, S)$ has a Hamilton decomposition.

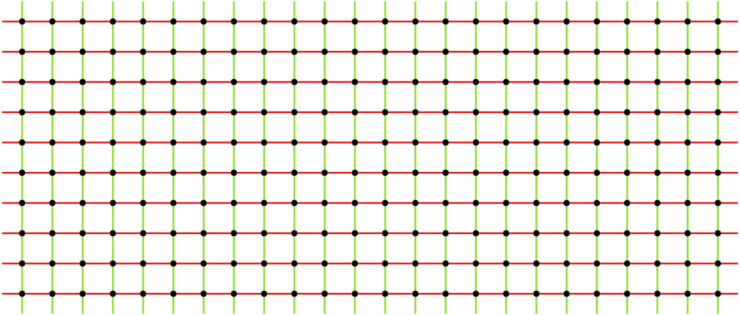
Erde, FL, Pitz '18+

Proof sketch

- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent

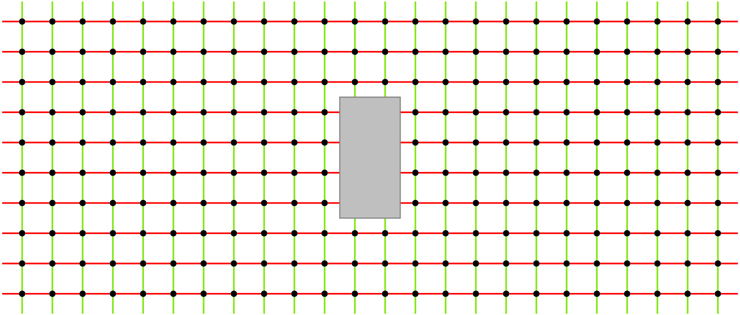
Proof sketch

- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



Proof sketch

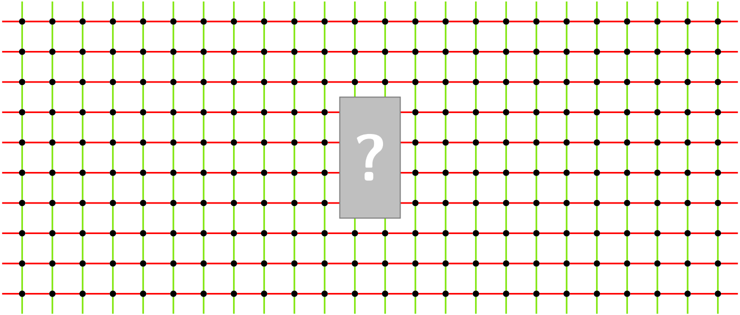
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes

Proof sketch

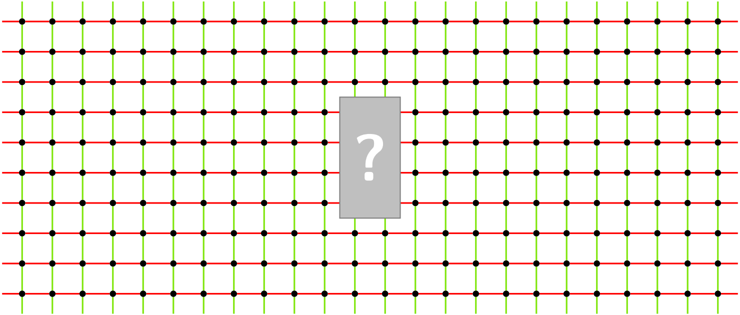
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes

Proof sketch

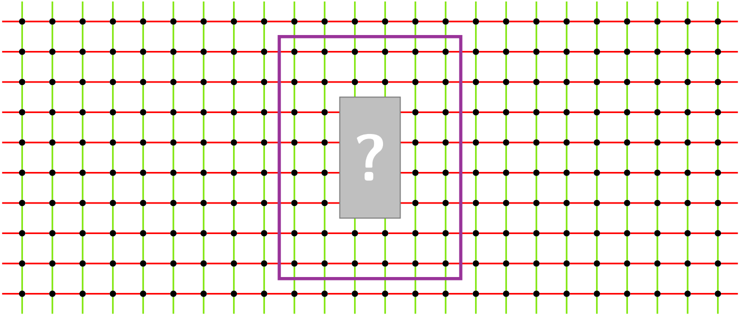
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

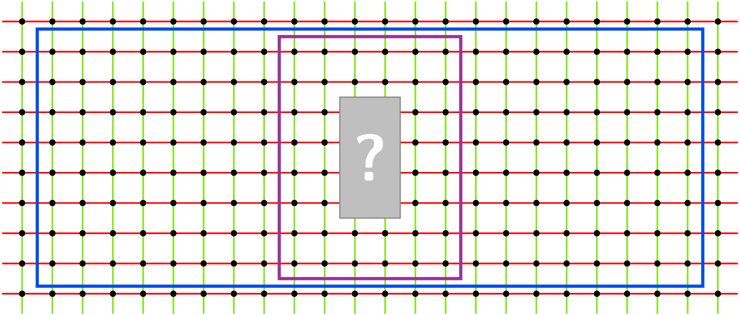
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

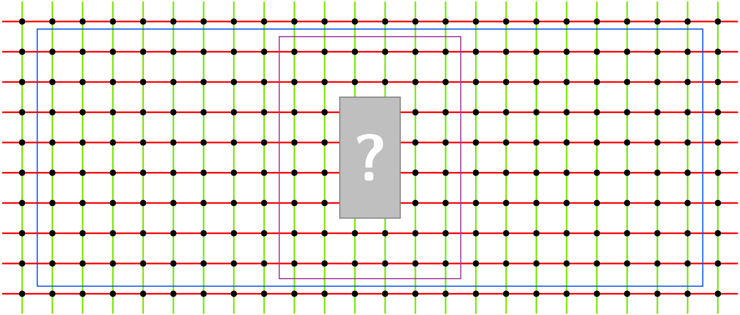
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

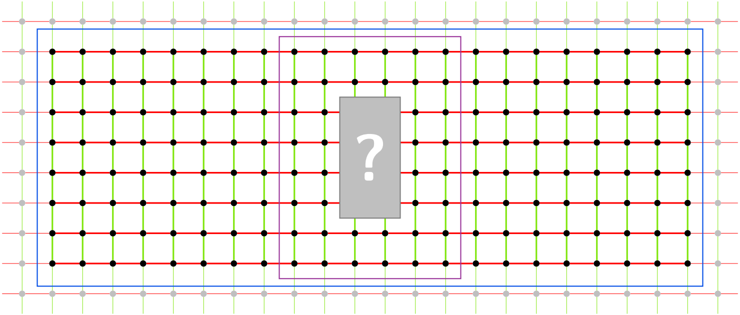
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

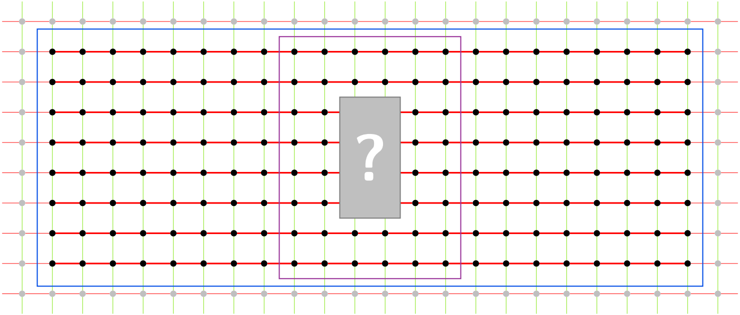
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

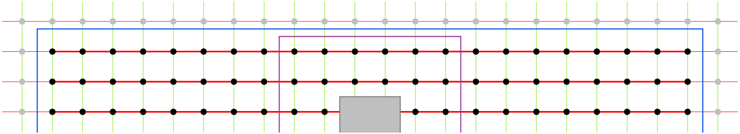
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



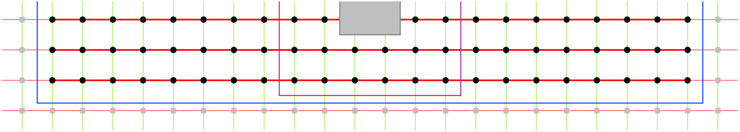
- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



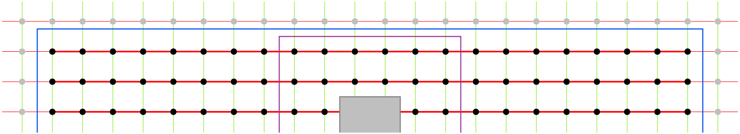
Definition. A colour flip at a square is what you expect it to be.



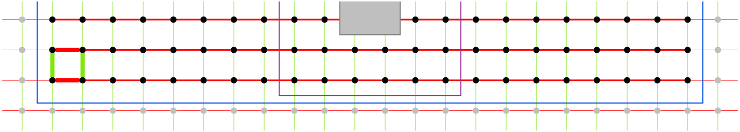
- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



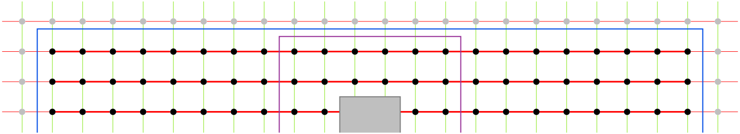
Definition. A colour flip at a square is what you expect it to be.



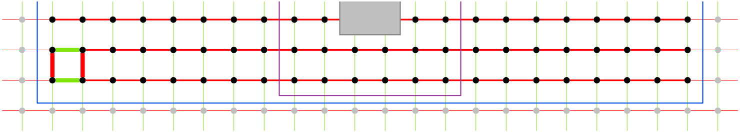
- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



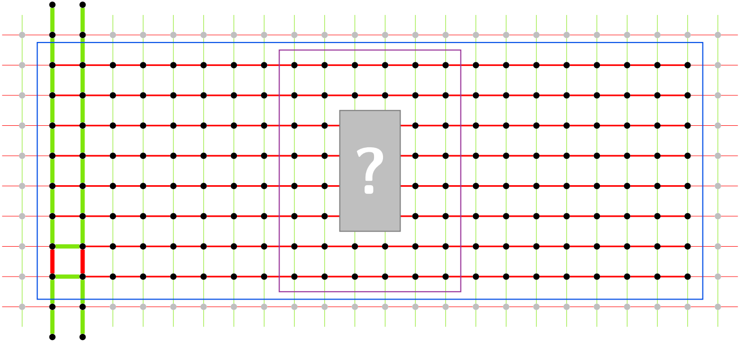
Definition. A colour flip at a square is what you expect it to be.



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

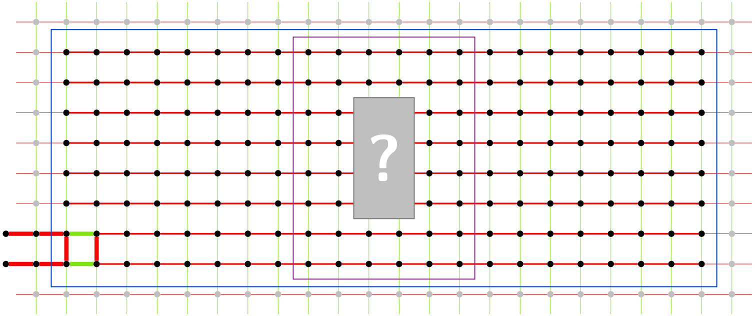
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

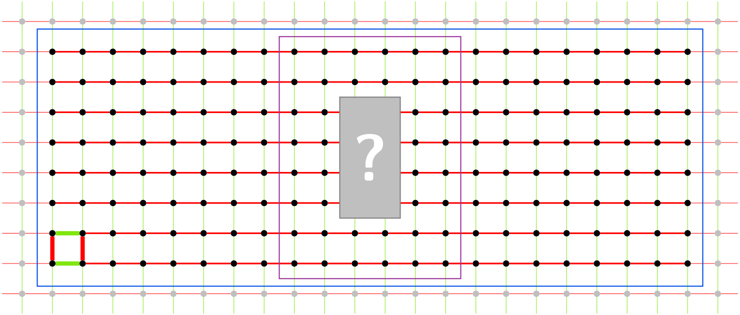
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

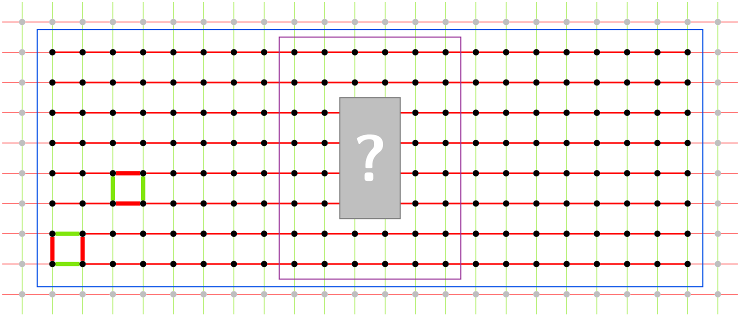
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

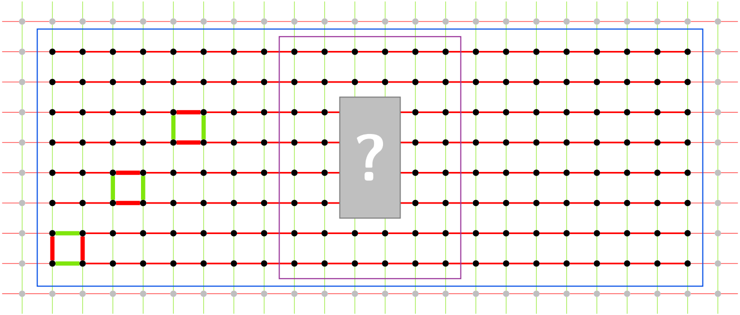
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

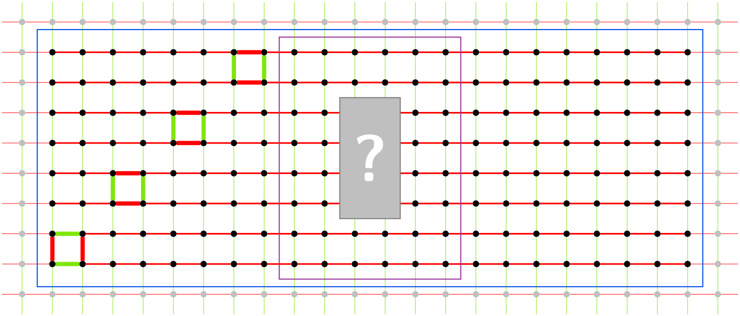
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

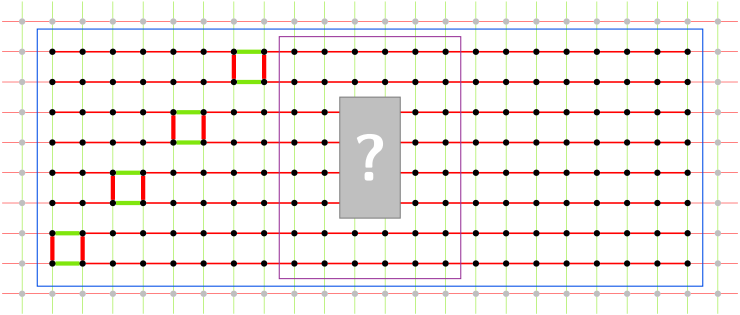
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

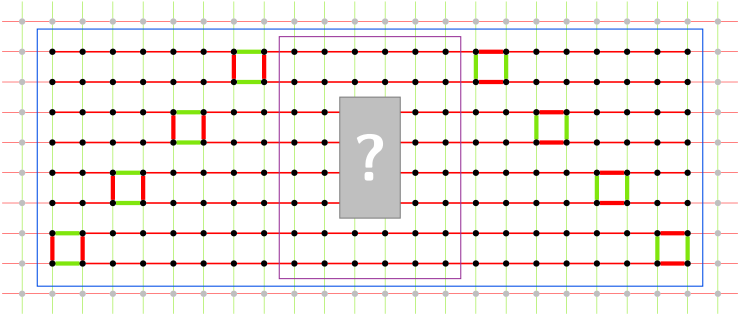
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

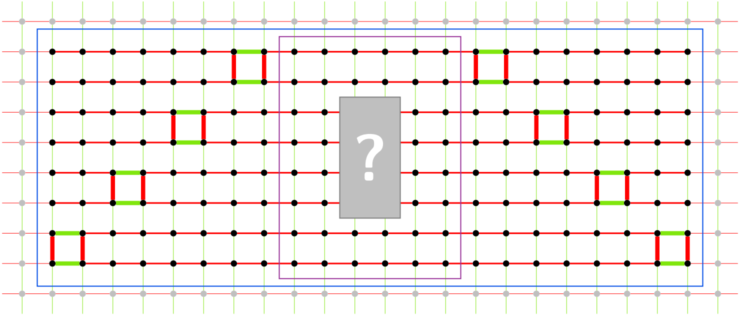
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

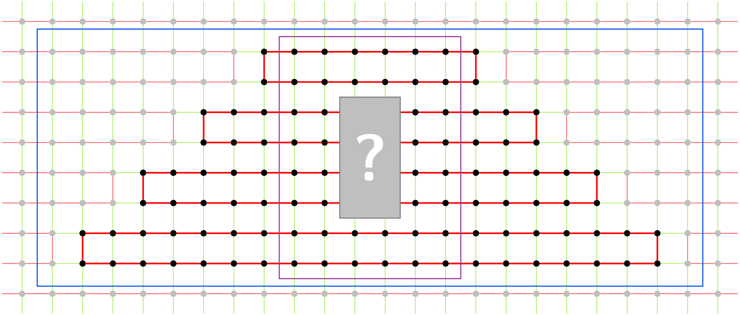
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

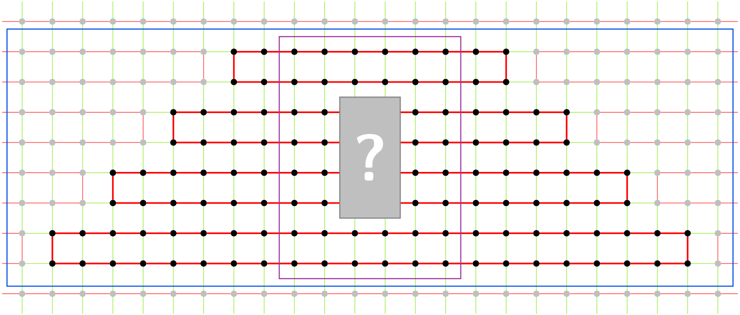
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

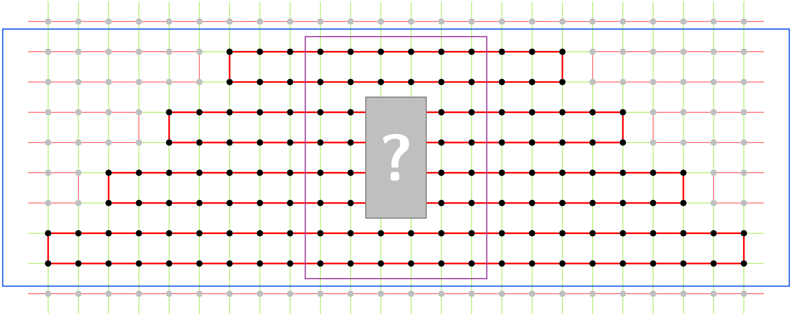
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

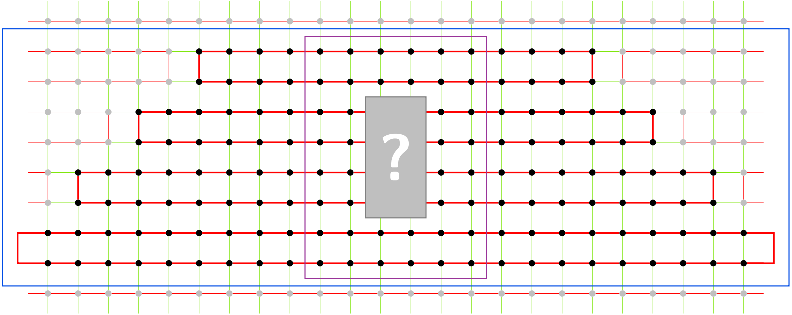
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

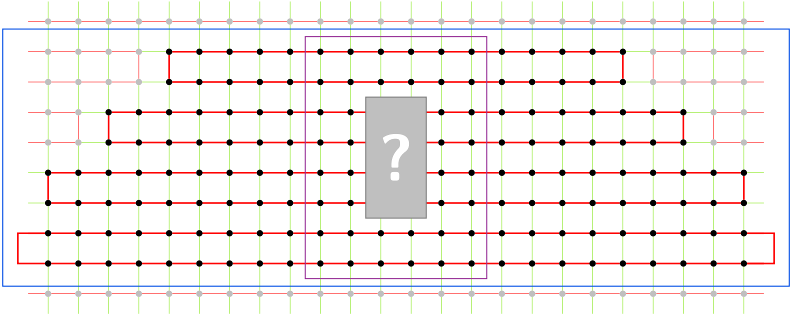
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

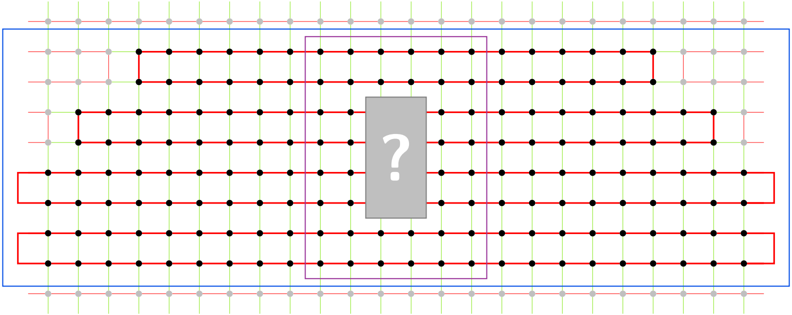
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

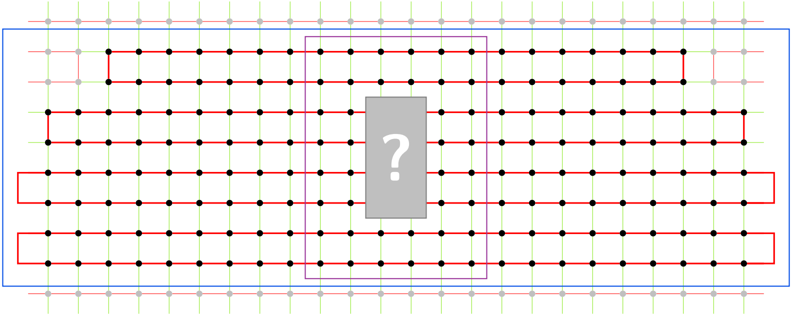
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

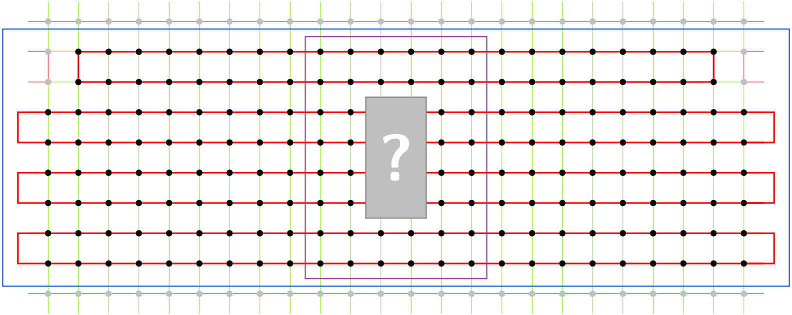
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

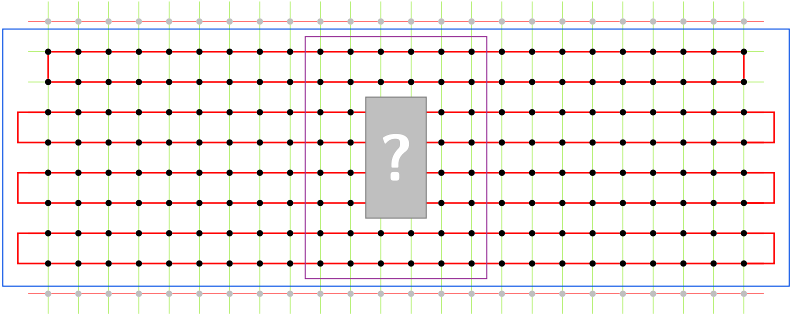
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

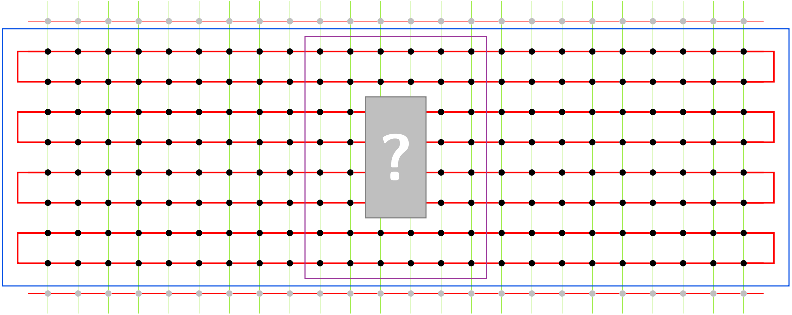
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

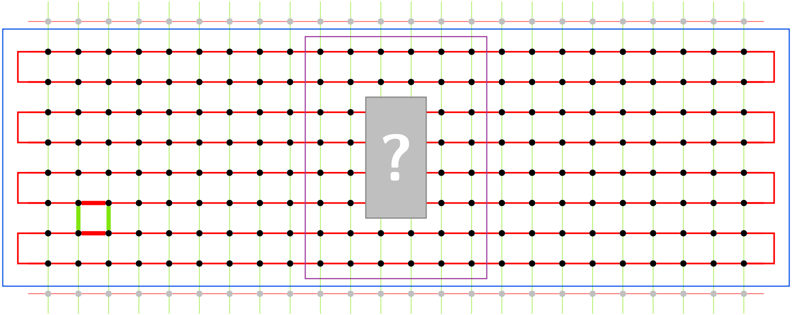
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

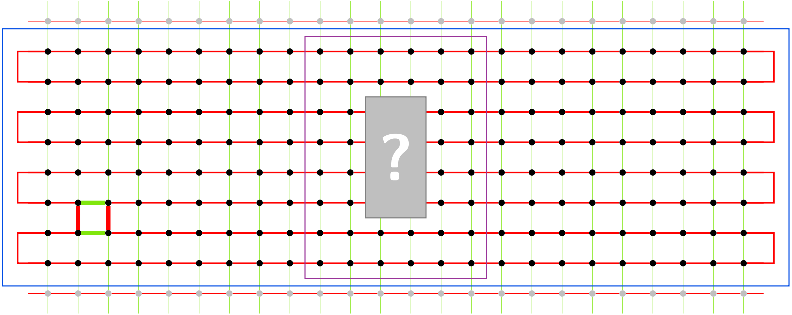
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

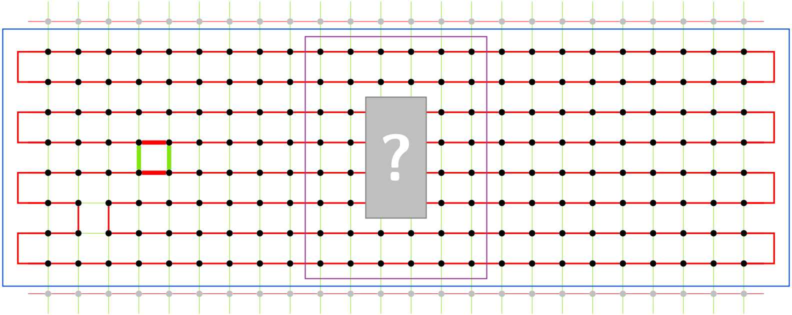
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

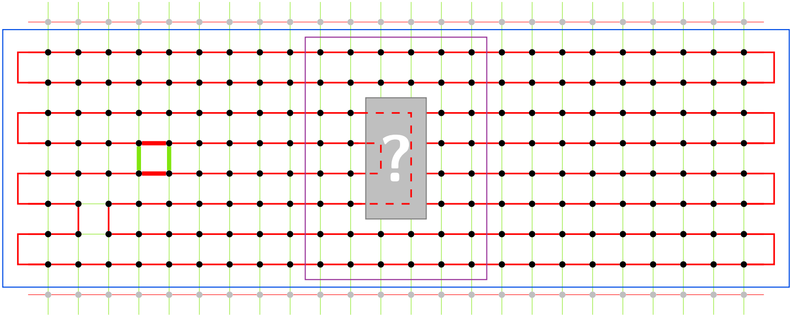
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

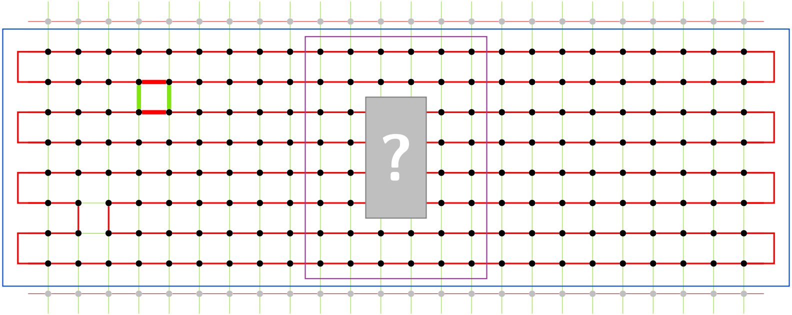
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

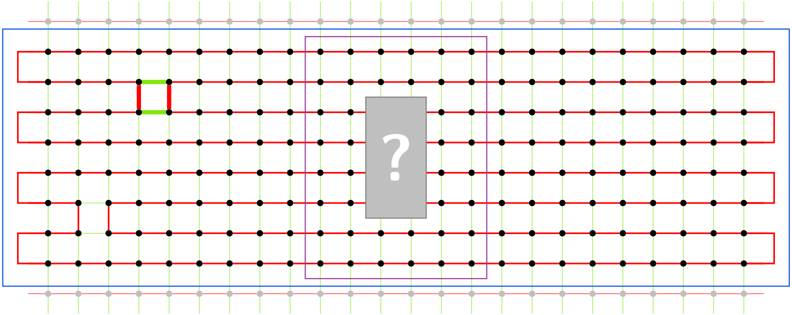
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

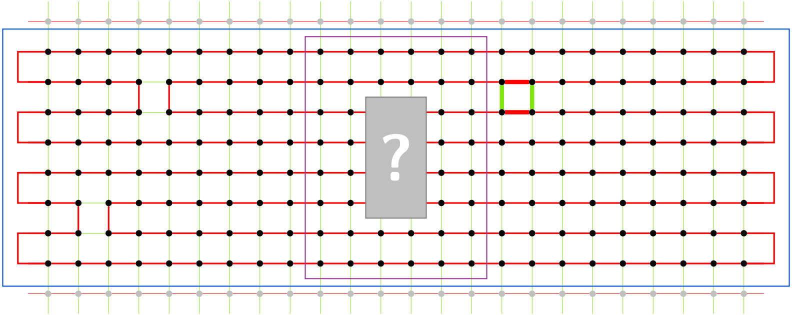
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

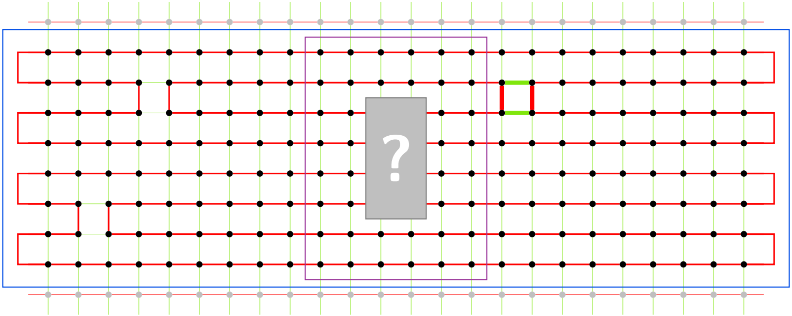
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

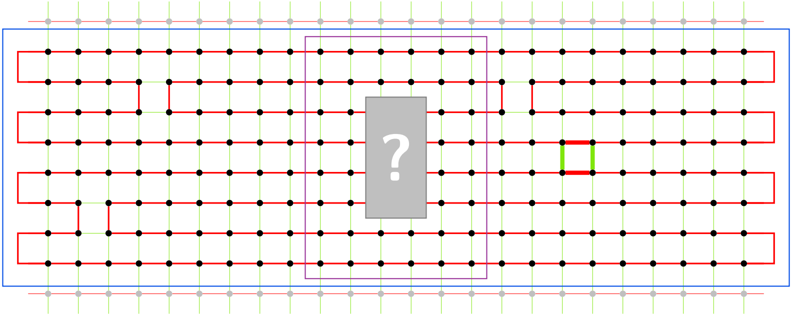
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

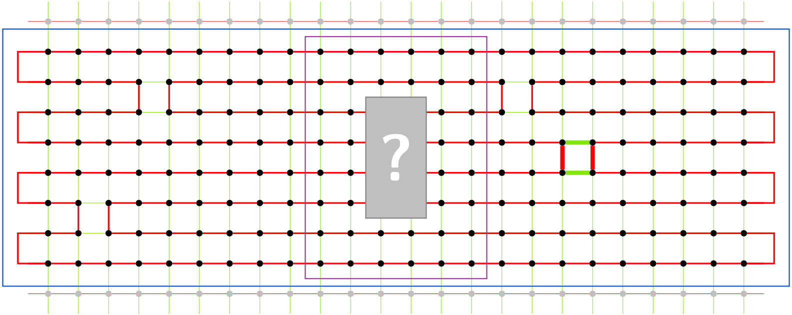
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

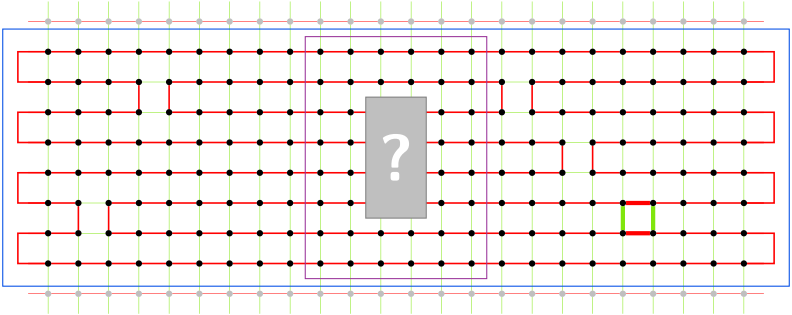
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

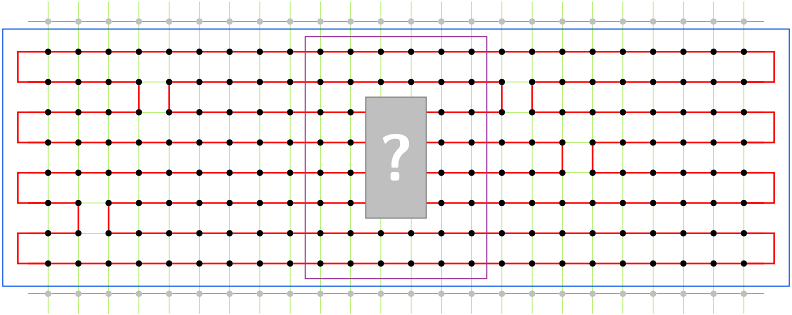
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

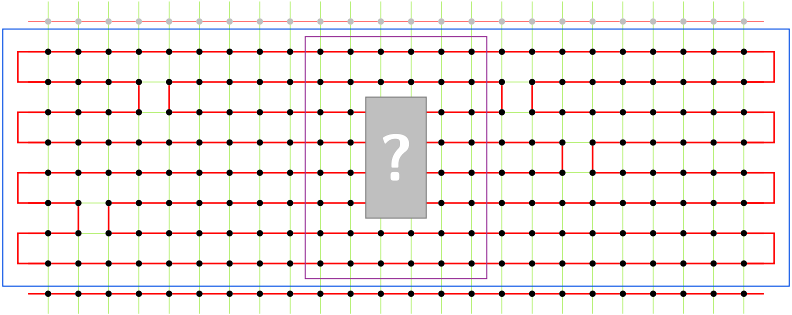
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

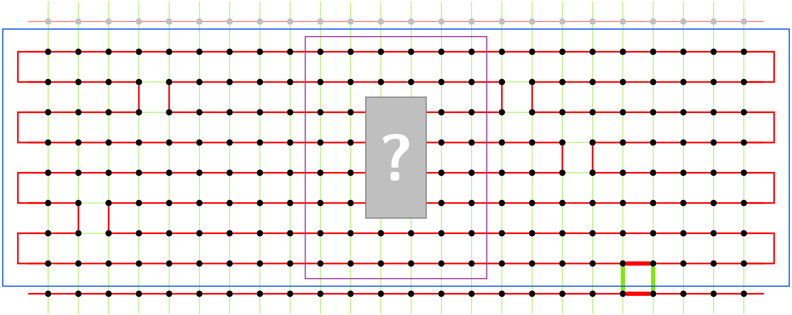
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

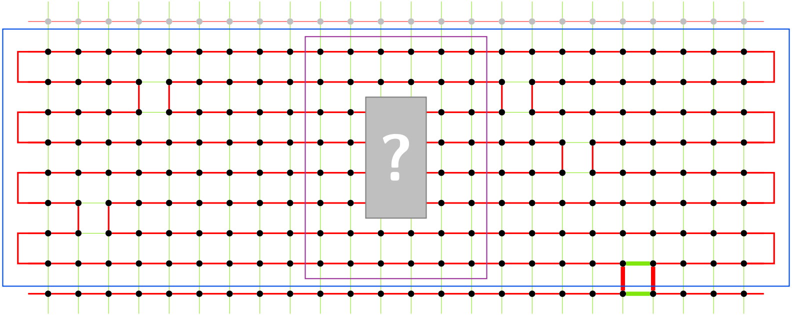
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

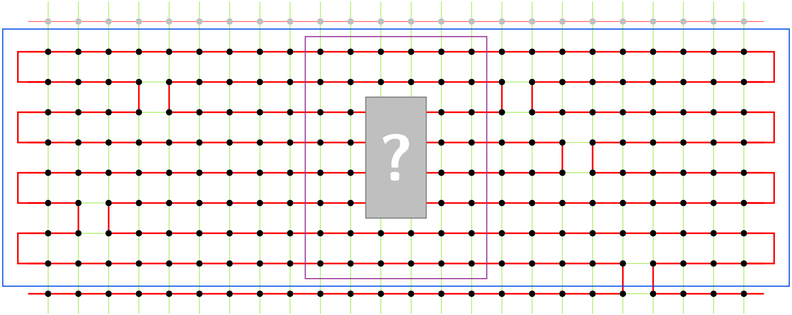
- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Proof sketch

- S a generating set without torsion elements,
- $g \in S$, $h \in S$ linearly independent



- standard colouring with finitely many changes
- all components in each colour are double rays

Final remarks

Theorem.
Let G be a one ended Abelian group, and let S be a generating set containing no torsion elements.
Then $\mathrm{Cay}(G, S)$ has a Hamilton decomposition.

Erde, FL, Pitz '18+

Final remarks

Theorem.
Let G be a one ended Abelian group, and let S be a generating set containing no torsion elements.
Then $\mathrm{Cay}(G, S)$ has a Hamilton decomposition.

Erde, FL, Pitz '18+

- What about torsion generators?
- Two-ended groups?

Final remarks

Theorem.
Let G be a one ended Abelian group, and let S be a generating set containing no torsion elements.
Then $\mathrm{Cay}(G, S)$ has a Hamilton decomposition.

Erde, FL, Pitz '18+

- What about torsion generators?
- Two-ended groups?
- Other notions of infinite Hamilton cycles?