Characterization of Generalized Petersen Graphs that are Kronecker Covers

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Graphs, groups and more:
celebrating Brian Alspach's 80th and Dragan Marušič's 65th birthdays

Outline

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Generalized Petersen graphs

- The family of generalized Petersen graphs, is well-known family of cubic graphs, introduced by Coxeter (1950) and named by Watkins (1969). They include Hamiltonian and non-Hamiltonian graphs, bipartite and non-bipartite graphs, vertex-transitive and non-vertex-transitive graphs, Cayley and non-Cayley graphs, arc-transitive graphs and non-arc transitive graphs, graphs of girth $3, 4, 5, 6, 7$ or 8.
- For $k < \frac{n}{2}$, $G(n, k)$ is a generalized Petersen graph on vertex set $\{u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}\}$. The edge set of $G(n, k)$ is formed by $u_i u_{i+1}$, $u_i v_i$ and v_i, v_{i+k} $(i \in \mathbb{Z}_n)$.
- Contains some famous members, such as Petersen graph $G(5, 2)$, n-prisms $G(n, 1)$, the Dürer graph $G(6, 2)$, the Möbius-Kantor graph $G(8, 3)$, the dodecahedron $G(10, 2)$, the Desargues graph $G(10, 3)$ and the Nauru graph $G(12, 5)$.

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Theorem (Alspach (1982))

The generalized Petersen graph $G(n,k)$ is Hamiltonian if and only if it is neither

1. *isomorphic to $G(n,2)$ with $n \equiv 5 \pmod 6$, nor*
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Let $G(n,k)$ be a generalized Petersen graph. Then:

1. *it is symmetric if and only if* $(n,k) \in \{(4,1),(5,2),(8,3),(10,2),(10,3),(12,5),(24,5)\}$,
2. *it is vertex-transitive if and only if* $k^2 \equiv \pm 1 \pmod{n}$ *or if* $n = 10$ *and* $k = 2$,
3. *it is a Cayley graph if and only if* $k^2 \equiv 1 \pmod{n}$,
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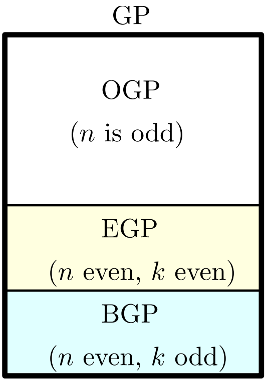
The family of generalized Petersen graphs is denoted by GP, and is further partitioned into three parts, depending on the parity of n and k. In particular,

- let $\mathrm{OGP} := \{G(n,k) \mid n \text{ is odd}\}$,
- let $\mathrm{BGP} := \{G(n,k) \mid n \text{ is even and } k \text{ is odd}\}$, and
- let $\mathrm{EGP} := \mathrm{GP} \setminus (\mathrm{OGP} \cup \mathrm{BGP})$.



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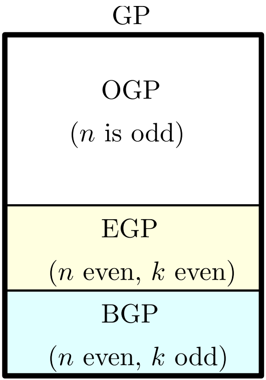

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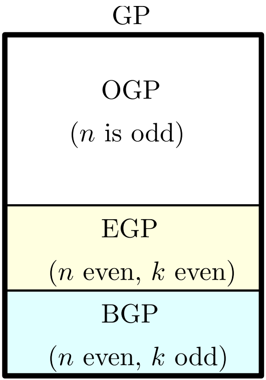

Outline

Kronecker double cover

- Also known as *bipartite double cover* or *canonical double cover* of G.
- Very natural definition; for a given graph G, its Kronecker double cover (denoted $\mathrm{DC}(G)$) is defined as tensor product $G \times K_2$.
- In other words: for a graph G, its double cover $\mathrm{DC}(G)$ can be defined as $V(\mathrm{DC}(G)) := \{v', v''\}_{v \in V(G)}$ with the edge-set $E(\mathrm{DC}(G)) := \{u'v'', u''v'\}_{uv \in E(G)}$.

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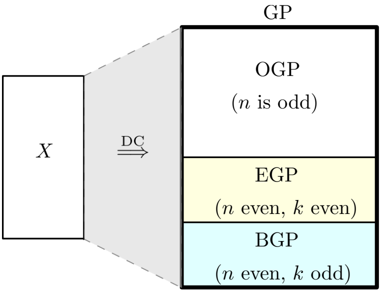
Basic questions

1. Which generalized Petersen graphs are Kronecker covers?
2. What is the structure of the corresponding quotients? Denote these by X.
3. Is $X \cap \mathrm{GP}$ empty, or what are its members ?
4. How does the planar members of X look like?



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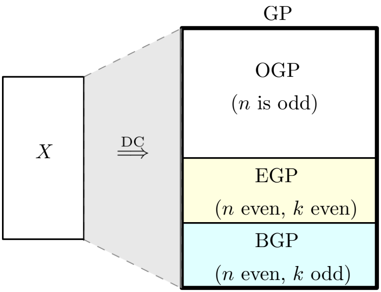

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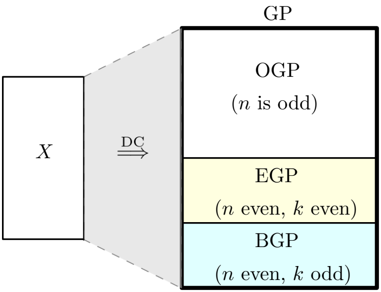

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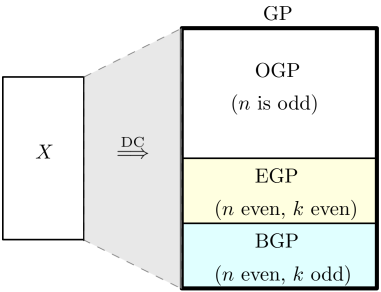

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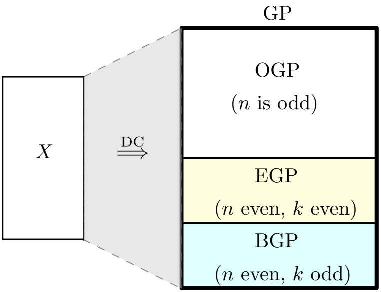

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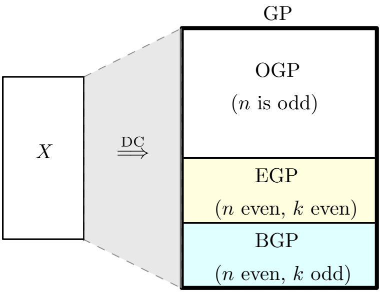

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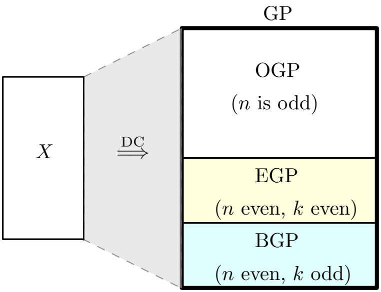
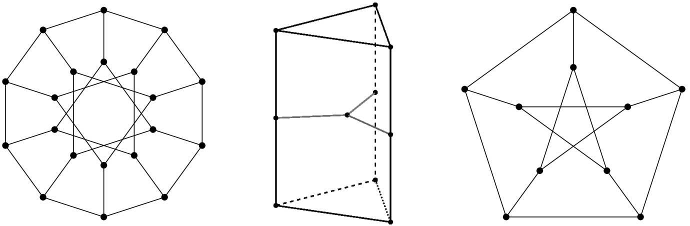


Our results

Theorem

Among the members of the family of generalized Petersen graphs, $G(10,3)$ is the only graph that is the Kronecker cover of two non-isomorphic graphs, the Petersen graph and $\mathcal{H}$ (the middle graph below). For any other $G \simeq G(n,k)$ (with $k < \frac{n}{2}$), it is a Kronecker cover if and only if:

1. *If $n \equiv 2 \pmod 4$ and k is odd, or*
2. *if $n \equiv 0 \pmod 4$ and k is odd, and $n \mid \frac{k^2-1}{2}$.*



Structure of quotients

How does members of X look like?

Definition

Assuming all numbers are modulo n, using LCF notation for cubic Hamiltonian graphs, define

$$C^{+}(n,k) = \left[\frac{n}{2}, \frac{n}{2} + (k-1), \frac{n}{2} + 2(k-1), \ldots, \frac{n}{2} + (n-1)(k-1)\right],$$

and similarly

$$C^{-}(n,k) = \left[\frac{n}{2}, \frac{n}{2} - (k+1), \frac{n}{2} - 2(k+1), \ldots, \frac{n}{2} - (n-1)(k+1)\right].$$

Structure of quotients

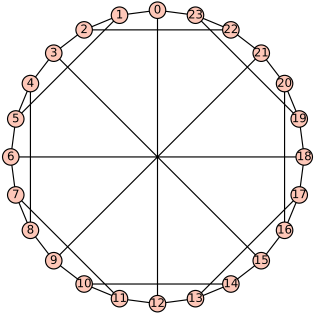


Figure: The graph $C^{-}(24, 7)$. As we observe later, $C^{-}(24, 7)$ is a quotient of $G(24, 7)$.

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1. *If $n \equiv 2 \pmod 4$ and k is odd, G is a Kronecker cover. In particular*
    1. *if $k < n/4$, the corresponding quotient graph is $G(n/2,k)$, and*
    2. *if $n/4 < k < n/2$ the quotient graph is $G(n/2, n/2-k)$.*
2. *If $n \equiv 0 \pmod 4$ and k is odd, G is a Kronecker cover if and only if $n \mid (k^2-1)/2$ and $k < n/2$. Moreover,*
    1. *if $k = 4t+1$ the corresponding quotient is $C^{+}(n,k)$ while*
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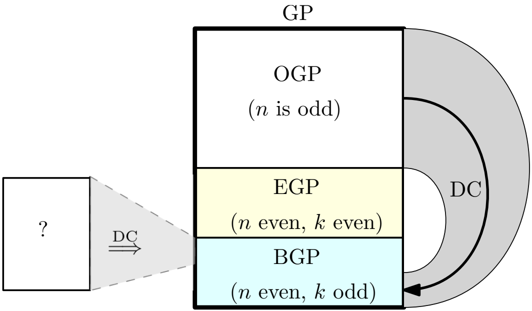
Petersen graphs, that are Kronecker covers

It is enough to investigate the quotients of BGP!



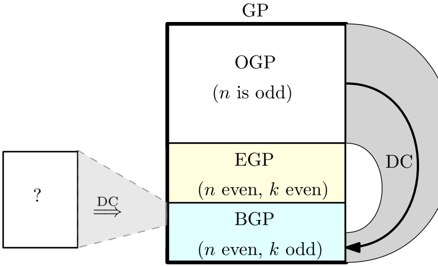
Petersen graphs, that are Kronecker covers

Preliminaries

Theorem (Imrich and Pisanski (2008))

Let G be a bipartite graph with bipartitions B_1 and B_2. There exists G' such that $\mathrm{DC}(G') \simeq G$, if and only if there exists an involution $\gamma \in Aut(G)$ such that $\gamma[B_1] = B_2$, and γ does not fix any edge.

It is hence enough to find all such involutions from the corresponding automorphism group – we will call these *Kronecker involutions*.



Petersen graphs, that are Kronecker covers

Preliminaries

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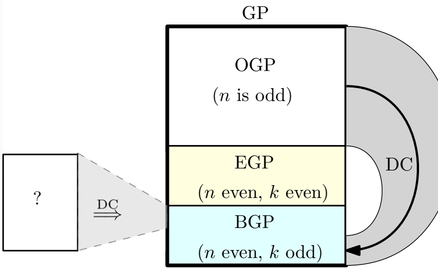

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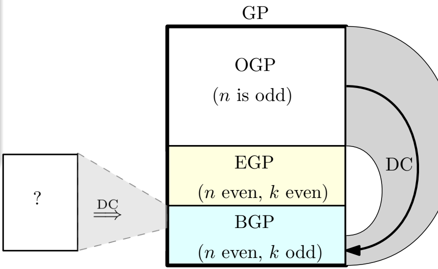

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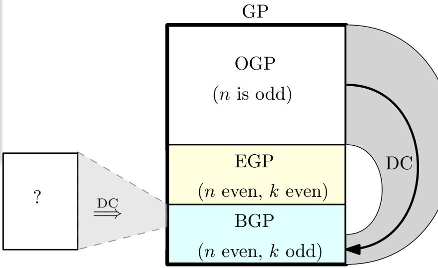

Petersen graphs, that are Kronecker covers

Preliminaries

Definition

For $i \in [0, n-1]$, define the permutations α, β and γ on $V(G(n,k))$ by

$$\begin{aligned} \alpha(u_i) &= u_{i+1}, & \alpha(v_i) &= v_{i+1}, \\ \beta(u_i) &= u_{-i}, & \beta(v_i) &= v_{-i}, \\ \gamma(u_i) &= v_{ki}, & \gamma(v_i) &= u_{ki}. \end{aligned}$$

Also, let $A(n,k)$ be the automorphism group of $G(n,k)$.

Petersen graphs, that are Kronecker covers

The automorphism group of $G(n, k)$

Theorem (Frucht et al. (1971))

If (n, k) is not one of $(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5)$, or $(24, 5)$, then the following holds:

- *if $k^2 \equiv 1 \mod n$, then*

$$A(n, k) = \langle \alpha, \beta, \gamma | \alpha^n = \beta^2 = \gamma^2 = 1, \alpha\beta = \beta\alpha^{-1}, \alpha\gamma = \gamma\alpha^k, \beta\gamma = \gamma\beta \rangle$$

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  In this case $\beta = \gamma^2$.

- *In all other cases the graph $G(n, k)$ is not vertex-transitive and*

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Proof of Case 1 of the Theorem.

Corollary

Let α, β, γ be the generators from above. Then, for any $\gamma \in \langle \alpha, \beta, \gamma \rangle$, we can find integers a, b, c such that $\gamma = \alpha^a \beta^b \gamma^c$.

Lemma

The following statements hold:

1. α^a *is a Kronecker involution iff.* $a = n/2$ *and* $n \equiv 2 \pmod 4$*;*
2. $\alpha^a \beta$ *is not a Kronecker involution;*
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- notice that $c \neq 0$ and $k^2 \equiv 1 \pmod n$,
- hence it is enough to consider the cases $\gamma = \alpha^a \gamma$ and $\gamma = \alpha^a \beta \gamma$.

At the same time, we need to assure

- that γ switches both bipartitions set-wise,
- that $\gamma^2 = \mathrm{id}$, and
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Proof for Case 2 of the Theorem, with $\gamma = \alpha^a\gamma$.

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Define Q, such that $k^2 - 1 = Qn$, let $a_{\min} = \frac{n}{\gcd(n,k+1)}$ and let $\alpha^a\gamma$ be a Kronecker involution. Then:

1. $\alpha^{2a}\gamma$ *is not a Kronecker involution;*
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Wrapping up the proof

- For any member of GP we identified all the Kronecker involutions
- In several cases, we have several of them!
- Unique quotient? Yes!

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Outline

Non-Planar Members of $X \setminus \mathrm{OGP}$

Other non-planar members of X include Möbius ladders, and some other interesting graphs...

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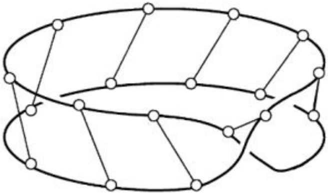

Figure: For $n \equiv 0 \pmod 4$, Kronecker cover of a Möbius ladder $C^{+}(n, 1)$, with $f(x) = \frac{n}{2} + x$, corresponds to $G(n, 1)$.

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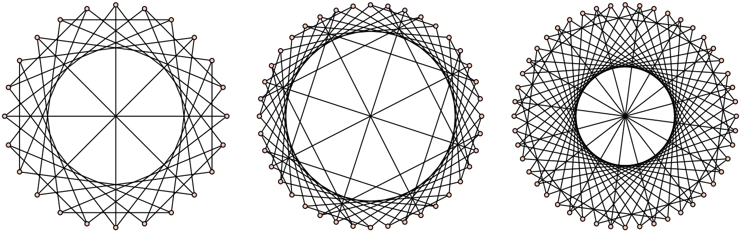

$DC(G) \simeq G(24, 7)$ $DC(G) \simeq G(40, 9)$ $DC(G) \simeq G(48, 17)$

Figure: Some other members of $C^+(n, k)$ and $C^-(n, k)$.

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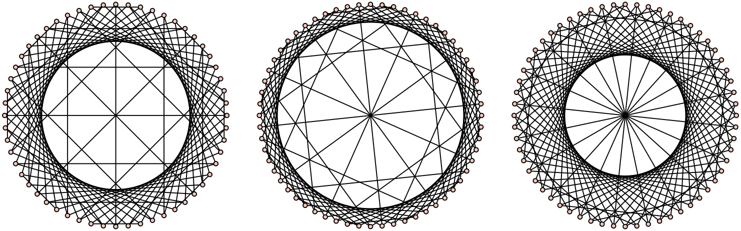

$DC(G) \simeq G(56, 15)$ $DC(G) \simeq G(60, 11)$ $DC(G) \simeq G(60, 19)$

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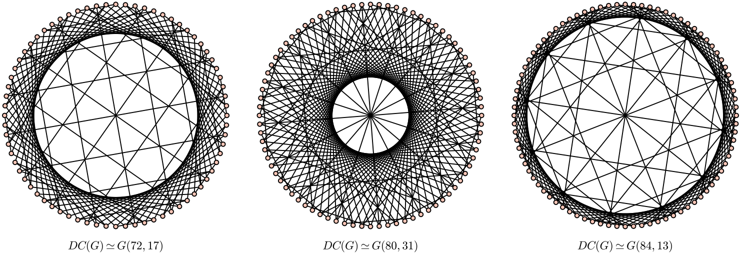

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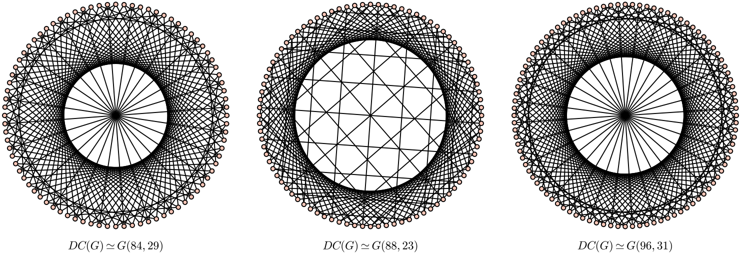

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Planar Members of $X \setminus \mathrm{OGP}$

Planar non-Petersen members of X are (maybe) nice!

- As far as computer calculation goes, the planar members of $X \setminus \mathrm{OGP}$ have a nice structure - their double covers correspond to the family $G(8i+4, 4i+1)$.

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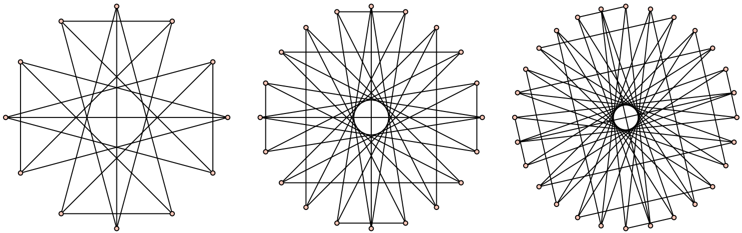

$DC(G) \simeq G(12,5)$ $DC(G) \simeq G(20,9)$ $DC(G) \simeq G(28,13)$

Figure: Kronecker cover of these graphs are members of GP.

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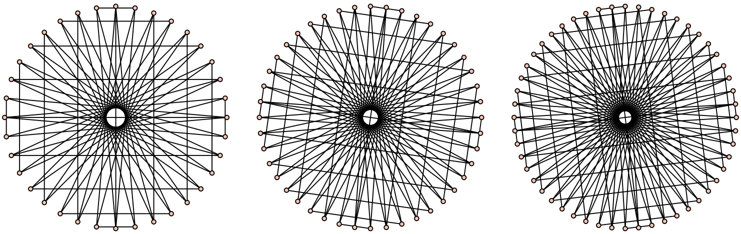

$DC(G) \simeq G(36, 17)$ $DC(G) \simeq G(44, 21)$ $DC(G) \simeq G(52, 25)$

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Planar Members of $X \setminus \mathrm{OGP}$

The corresponding covers seems to be embeddings of honeycomb on torus.

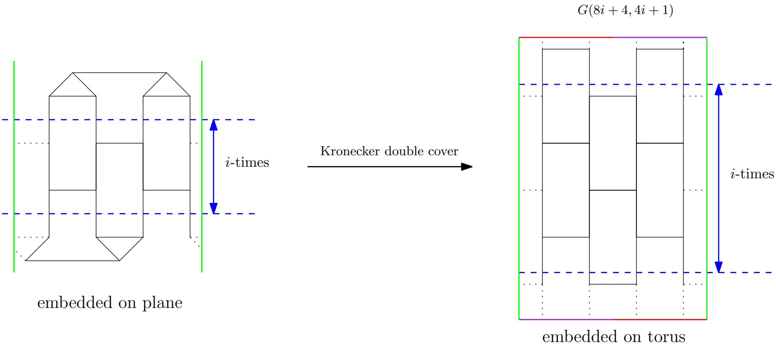

Figure: Honeycomb embedding of $G(8i+4, 4i+1)$ on a torus. Note that the quotient corresponds to a planar graph $C^{+}(8i+4, 4i+1)$.

Outlook

- We determined the members of GP which are Kronecker covers.
- The corresponding quotients are described by two special families $C^+(n, k)$ and $C^-(n, k)$.

We plan to:

- look into some topological properties of $C^+(n, k)$ and $C^-(n, k)$, and corresponding properties.

One can show that all Generalized Petersen graphs which admit a honeycomb embedding to torus are Cayley. In addition, if such a graph is a Kronecker cover, then its quotient is planar and of type $C^+(8i + 4, 4i + 1)$.

Conjecture

Every planar quotient of GP is either from GP, or is of type $C^+(8i + 4, 4i + 1)$.

Happy birthdays!