Hamilton Paths and Cycles in Vertex-transitive Graphs

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Koper, June 1, 2018

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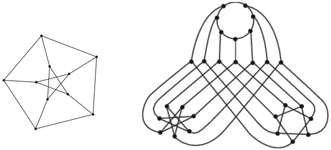
Lovász question

Lovász, 1969

Does every connected vertex-transitive graph have a Hamilton path?

Hamiltonicity of vertex - transitive graphs

All known VTG have Hamilton path. Only 4 CVTG (having at least three vertices) not having a Hamilton cycle are known to exist. None of them is a Cayley graph.



Folklore conjecture

Every connected Cayley graph contains a Hamilton cycle.

The current situation

Hamilton cycles (paths) are known to exist in these cases:

- VTG of order p, $2p$, $3p$, $4p$, $5p$, $6p$, $2p^2$, p^k (for $k \leq 4$), $10p$ ($p > 7$) (Alspach, Chen, Du, Marušič, Parsons, Šparl, Zhang, KK, etc.);
- CG of p-groups (Witte);
- VTG having groups with a cyclic commutator subgroup of order p^k (Durenberger, Gavlas, Keating, Marušič, Morris, Morris-Witte, etc.).
- CG $\mathrm{Cay}(G, \{a, b, a^b\})$, where a is an involution (Pak, Radoičić).
- Cubic CG $\mathrm{Cay}(G, S)$, where $S = \{a, b, c\}$ and $a^2 = b^2 = c^2 = 1$ and $ab = ba$ (Cherkassoff, Sjerve).
- Cubic CG $\mathrm{Cay}(G, S)$, where $S = \{a, x, x^{-1}\}$ and $a^2 = 1$, $x^s = 1$ and $(ax)^3 = 1$ (Glover, Marušič).
- CG on groups whose orders have small prime factorization (Morris-Witte, ...).
- and in some other cases.

In short: the problem is still open.

Hamiltonicity of vertex-transitive graphs

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ScienceDirect
European Journal of Combinatorics 29 (2008) 423–438
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Hamiltonicity of vertex-transitive graphs of order $4p$

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Received 26 September 2006; accepted 15 February 2007
Available online 1 March 2007

J Algebr Comb
DOI 10.1007/s10801-009-0172-5

Hamiltonian cycles in cubic Cayley graphs: the $\langle 2, 4k, 3\rangle$ case

Henry H. Glover · Klavdija Kutnar · Dragan Marušič

Received: 22 May 2008 / Accepted: 16 February 2009

Discrete Mathematics 309 (2009) 5444–5460
Contents lists available at ScienceDirect
Discrete Mathematics
journal homepage: www.elsevier.com/locate/disc

Hamilton paths and cycles in vertex-transitive graphs of order $6p$

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Hamiltonicity of vertex-transitive graphs

dMFA
ARS MATHEMATICA CONTEMPORANEA
Also available at http://amc.imfm.si
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 5 (2012) 27–71

Hamiltonian cycles in Cayley graphs whose order has few prime factors

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Received 3 October 2010, accepted 4 April 2011, published online xx October 2011

European Journal of Combinatorics 33 (2012) 1043–1077
Contents lists available at SciVerse ScienceDirect
European Journal of Combinatorics
journal homepage: www.elsevier.com/locate/ejc
ELSEVIER

Hamilton paths in vertex-transitive graphs of order $10p$

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Proceedings of the London Mathematical Society Advance Access published January 29, 2012
Proc. London Math. Soc. Page 1 of 27

doi:10.1112/plms/pdr042

Hamilton cycles in $(2, \mathrm{odd}, 3)$-Cayley graphs

Henry H. Glover, Klavdija Kutnar, Aleksander Malnič and Dragan Marušič

Hamiltonicity of vertex-transitive graphs

Discrete Mathematics 78 (1989) 25–36
North-Holland

LIFTING HAMILTON CYCLES OF QUOTIENT GRAPHS

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Dedicated to the memory of Tory Parsons

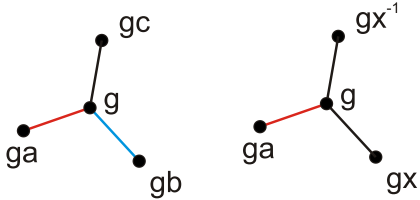
Received 19 November 1987
Revised 20 August 1988

Using a standard notion of a quotient graph of certain vertex-transitive graphs, methods for lifting a Hamilton cycle in the quotient graph to a Hamilton cycle in the original graph are discussed.

Cubic Cayley graphs

If $\mathrm{Cay}(G,S)$ is a cubic Cayley graph then $|S| = 3$, and either

- $S = \{a, b, c \mid a^2 = b^2 = c^2 = 1\}$, or
- $S = \{a, x, x^{-1} \mid a^2 = x^s = 1\}$ where $s \geq 3$.



$(2, s, t)$-Cayley graphs

Definition

Let $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \ldots \rangle$ be a group. Then the Cayley graph $X = \mathrm{Cay}(G, S)$ of G with respect to the generating set $S = \{a, x, x^{-1}\}$ is called $(2, s, t)$-Cayley graph.

A $(2, s, t)$-Cayley graph is a Cayley graph of a quotient group of the triangle group $T(2, s, t)$.

A $(2, s, 3)$-Cayley graph is a Cayley graph of a quotient group of the modular group $\langle a, x \mid a^2 = (ax)^3 = 1\rangle$.

Hamiltonicity of $(2, s, 3)$-Cayley graphs

Glover, Marušič, J. Eur. Math. Soc., 2007

Let $s \geq 3$ be an integer and let $X = \mathrm{Cay}(G, S)$ be a $(2, s, 3)$-Cayley graph of a group G. Then X has

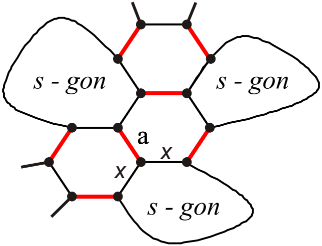
- a Hamilton cycle when $|G|$ is congruent to 2 modulo 4, and
- a cycle of length $|G| - 2$, and also a Hamilton path, when $|G|$ is congruent to 0 modulo 4.

Essential ingredients in the proof

A $(2, s, 3)$-Cayley graph X can be embedded in the closed orientable surface of genus

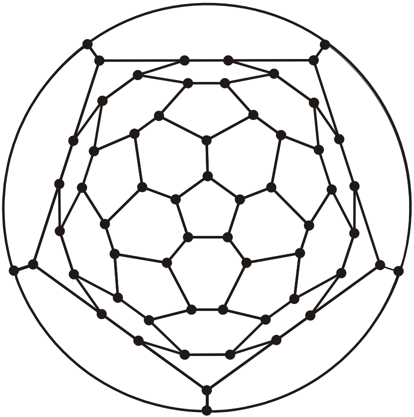
$$1 + (s - 6)|G|/12s$$

with faces $|G|/s$ disjoint s-gons and $|G|/3$ hexagons.



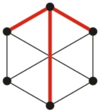
Soccer ball

(2, 5, 3)-Cayley graph of $A_5 = \langle a, x \mid a^2 = x^5 = (ax)^3 = 1 \rangle$.

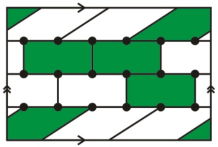


Example: $|G| \equiv 2 (\mathrm{mod}\ 4)$

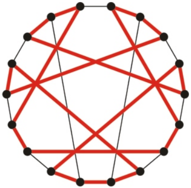
$G = S_3 \times \mathbb{Z}_3$ with a $(2,6,3)$-presentation $\langle a, x \mid a^2 = x^6 = (ax)^3 = 1, \ldots \rangle$, where $a = ((12), 0)$ and $x = ((13), 1)$.



The corresponding hexagon graph.



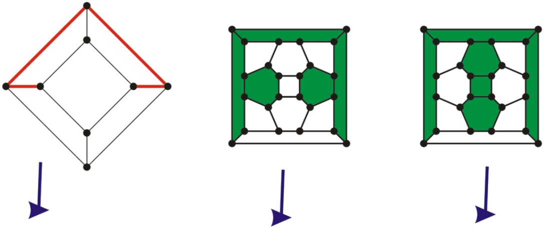
A Hamilton tree of hexagons.



The corresponding Hamilton cycle in X.

Example: $|G| \equiv 0 (\mathrm{mod}\ 4)$

$G = S_4$ with a $(2,4,3)$-presentation $\langle a, x \mid a^2 = x^4 = (ax)^3 = 1 \rangle$, where $a = (12)$ and $x = (1234)$.

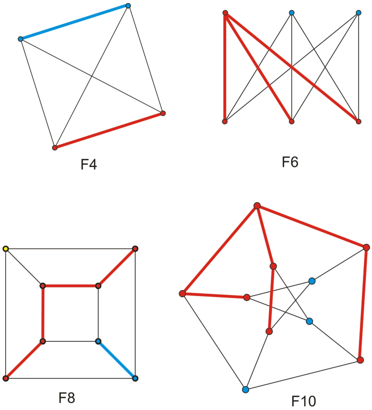


The same tree in the corresponding hexagon graph

A tree of hexagons, whose boundary is a cycle missing only two vertices in the spherical Cayley map of X.

A modified tree of faces (including also a square).

Cyclically stable subsets



Payan, Sakarovitch, 1975

Payan, Sakarovitch, 1975

Let X be a cyclically 4-edge-connected cubic graph of order n, and let S be a maximum cyclically stable subset of $V(X)$. Then $|S| = \lfloor (3n-2)/2 \rfloor$ and more precisely, the following hold.

- If $n \equiv 2\,(mod\,4)$ then $|S| = (3n-2)/4$, and $X[S]$ is a tree and $V(X) \setminus S$ is an independent set of vertices;
- If $n \equiv 0\,(mod\,4)$ then $|S| = (3n-4)/4$, and either $X[S]$ is a tree and $V(X) \setminus S$ induces a graph with a single edge, or $X[S]$ has two components and $V(X) \setminus S$ is an independent set of vertices.

Nedela, Škoviera, 1995

Nedela, Škoviera, 1995

The edge cyclic connectivity of a cubic vertex-transitive graph X equals its girth $g(X)$.

$(2, s, 3)$-Cayley graphs of order $0 \pmod 4$

For a $(2, s, 3)$-Cayley graph of order $0 \pmod 4$ three cases can occur:

- $s \equiv 0 \pmod 4$.
- $s \equiv 2 \pmod 4$.
- s odd.

Hamiltonicity of $(2, s, 3)$-Cayley graphs, $s \equiv 0 \pmod 4$

Glover, KK, Marušič, 2009

Let $s \equiv 0 \pmod 4 \geq 4$ be an integer. Then a $(2, s, 3)$-Cayley graph X has a Hamilton cycle.

Essential ingredients in the proof

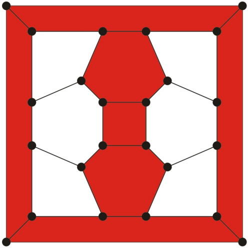
- Glover-Marušič method.
- Classification of cubic ATG of girth 6.
- Results on cubic ATG admitting a 1-regular subgroup.

Modification process in $(2, 4k, 3)$ case

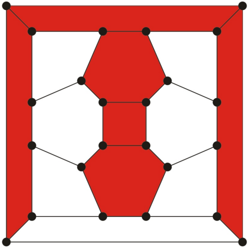
We consider the graph of hexagons Hex$X$.

The graph of hexagons Hex$X$ is modified so as to get a graph Mod$X$ with $2 \equiv \pmod 4$ vertices and cyclically 4-edge-connected, so to be able to get by [Payan, Sakarovitch, 1975] a tree whose complement is an independent set of vertices giving rise to a Hamilton cycle in X.

Example



Modified Hex surface for S_4 <2,4,3>



Hamilton tree of faces for S_4 <2,4,3>

Hamiltonicity of $(2, s, 3)$-Cayley graphs, s odd

Glover, KK, Malnič, Marušič, 2012

Let s be an odd integer. Then a $(2, s, 3)$-Cayley graph X has a Hamilton cycle.

- $\langle x \rangle$ is corefree in $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle$:

  A method similar to the method used in $s \equiv 0 \pmod 4$ case gives us a Hamilton cycle as a boundary of a Hamilton tree of faces consisting of hexagons and two s-gons.

- $\langle x \rangle$ is not corefree in $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle$:

  Results about lifts of Hamilton cycles in covers of graphs are needed.

Hamiltonicity of $(2, s, 3)$-Cayley graphs, s odd

Hamiltonicity of $(2, s, t)$-Cayley graphs

X ... $(2, s, t)$-Cayley graph

X_{2t} ... $2t$-gonal graph (arc-transitive graph admitting a 1-regular subgroup with cyclic vertex-stabilizer)

If the vertex set V of X_{2t} decomposes into $(I, V - I)$ with I independent set and $V - I$ induces a tree then X contains a Hamiltonian cycle.

Question: Characterize (tetravalent) arc-transitive graphs admitting a 1-regular subgroup via their decomposition properties.

In particular, characterize non-near-bipartite graphs amongst them.

8th PhD Summer School in Discrete Mathematics Rogla, July 1 - 7, 2018

Minicourse Lecturers:
Gabriel Verret, The University of Auckland, NZ
Colva Roney-Dougal, University of St Andrews, UK

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Thanks!