Hamilton Paths and Cycles in Vertex-transitive Graphs

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Lovász question

Lovász, 1969
Does every connected vertex-transitive graph have a Hamilton path?
All known VTG have Hamilton path. Only 4 CVTG (having at least three vertices) not having a Hamilton cycle are known to exist. None of them is a Cayley graph.

Folklore conjecture
Every connected Cayley graph contains a Hamilton cycle.
The current situation

Hamilton cycles (paths) are known to exist in these cases:

- VTG of order $p$, $2p$, $3p$, $4p$, $5p$, $6p$, $2p^2$, $p^k$ (for $k \leq 4$), $10p$ ($p > 7$) (Alspach, Chen, Du, Marušič, Parsons, Šparl, Zhang, KK, etc.);
- CG of $p$-groups (Witte);
- VTG having groups with a cyclic commutator subgroup of order $p^k$ (Durenberger, Gavlas, Keating, Marušič, Morris, Morris-Witte, etc.).
- CG Cay($G$, $\{a, b, a^b\}$), where $a$ is an involution (Pak, Radoičić).
- Cubic CG Cay($G$, $S$), where $S = \{a, b, c\}$ and $a^2 = b^2 = c^2 = 1$ and $ab = ba$ (Cherkassoff, Sjerve).
- Cubic CG Cay($G$, $S$), where $S = \{a, x, x^{-1}\}$ and $a^2 = 1$, $x^s = 1$ and $(ax)^3 = 1$ (Glover, Marušič).
- CG on groups whose orders have small prime factorization (Morris-Witte, ...).
- and in some other cases.

In short: the problem is still open.
Hamiltonicity of vertex-transitive graphs of order $4p$

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Hamiltonian cycles in cubic Cayley graphs: the $(2, 4k, 3)$ case

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Hamilton paths and cycles in vertex-transitive graphs of order $6p$

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Hamiltonian cycles in Cayley graphs whose order has few prime factors

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Hamilton paths in vertex-transitive graphs of order 10p
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Hamilton cycles in (2, odd, 3)-Cayley graphs

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LIFTING HAMILTON CYCLES OF QUOTIENT GRAPHS

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Dedicated to the memory of Tory Parsons

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Using a standard notion of a quotient graph of certain vertex-transitive graphs, methods for lifting a Hamilton cycle in the quotient graph to a Hamilton cycle in the original graph are discussed.
If Cay($G, S$) is a cubic Cayley graph then $|S| = 3$, and either

- $S = \{a, b, c \mid a^2 = b^2 = c^2 = 1\}$, or
- $S = \{a, x, x^{-1} \mid a^2 = x^s = 1\}$ where $s \geq 3$.
(2, s, t)-Cayley graphs

Definition

Let $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \ldots \rangle$ be a group. Then the Cayley graph $X = \text{Cay}(G, S)$ of $G$ with respect to the generating set $S = \{a, x, x^{-1}\}$ is called $(2, s, t)$-Cayley graph.

A $(2, s, t)$-Cayley graph is a Cayley graph of a quotient group of the triangle group $T(2, s, t)$.

A $(2, s, 3)$-Cayley graph is a Cayley graph of a quotient group of the modular group $\langle a, x \mid a^2 = (ax)^3 = 1 \rangle$. 
Let $s \geq 3$ be an integer and let $X = \text{Cay}(G, S)$ be a $(2, s, 3)$-Cayley graph of a group $G$. Then $X$ has

- a Hamilton cycle when $|G|$ is congruent to 2 modulo 4, and
- a cycle of length $|G| - 2$, and also a Hamilton path, when $|G|$ is congruent to 0 modulo 4.
A $(2, s, 3)$-Cayley graph $X$ can be embedded in the closed orientable surface of genus

$$1 + (s - 6)|G|/12s$$

with faces $|G|/s$ disjoint $s$-gons and $|G|/3$ hexagons.
Soccer ball

$(2, 5, 3)$-Cayley graph of $A_5 = \langle a, x \mid a^2 = x^5 = (ax)^3 = 1 \rangle$. 
Example: $|G| \equiv 2 \pmod{4}$

$G = S_3 \times \mathbb{Z}_3$ with a $(2, 6, 3)$-presentation
\[
\langle a, x \mid a^2 = x^6 = (ax)^3 = 1, \ldots \rangle,
\]
where $a = ((12), 0)$ and $x = ((13), 1)$. 

The corresponding hexagon graph.

A Hamilton tree of hexagons.

The corresponding Hamilton cycle in $X$. 

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Example: \(|G| \equiv 0(\text{mod } 4)\)

\[ G = S_4 \text{ with a } (2, 4, 3)\text{-presentation } \langle a, x \mid a^2 = x^4 = (ax)^3 = 1 \rangle, \]

where \(a = (12)\) and \(x = (1234)\).
Cyclically stable subsets

F4

F6

F8

F10
Let $X$ be a cyclically 4-edge-connected cubic graph of order $n$, and let $S$ be a maximum cyclically stable subset of $V(X)$. Then $|S| = \lfloor (3n - 2)/2 \rfloor$ and more precisely, the following hold.

- If $n \equiv 2 \pmod{4}$ then $|S| = (3n - 2)/4$, and $X[S]$ is a tree and $V(X) \setminus S$ is an independent set of vertices;
- If $n \equiv 0 \pmod{4}$ then $|S| = (3n - 4)/4$, and either $X[S]$ is a tree and $V(X) \setminus S$ induces a graph with a single edge, or $X[S]$ has two components and $V(X) \setminus S$ is an independent set of vertices.
The edge cyclic connectivity of a cubic vertex-transitive graph $X$ equals its girth $g(X)$. 
For a $(2, s, 3)$-Cayley graph of order $0 \pmod{4}$ three cases can occur:

- $s \equiv 0 \pmod{4}$.
- $s \equiv 2 \pmod{4}$.
- $s$ odd.
Let \( s \equiv 0 \pmod{4} \geq 4 \) be an integer. Then a \((2, s, 3)\)-Cayley graph \( X \) has a Hamilton cycle.

**Essential ingredients in the proof**

- Glover-Marušič method.
- Classification of cubic ATG of girth 6.
- Results on cubic ATG admitting a 1-regular subgroup.
We consider the graph of hexagons $\text{Hex}X$.

The graph of hexagons $\text{Hex}X$ is modified so as to get a graph $\text{Mod}X$ with $2 \equiv (\text{mod } 4)$ vertices and cyclically 4-edge-connected, so to be able to get by [Payan, Sakarovitch, 1975] a tree whose complement is an independent set of vertices giving rise to a Hamilton cycle in $X$. 
Example

Modified Hex surface for $S_4 <2,4,3>$

Hamilton tree of faces for $S_4 <2,4,3>$
Let $s$ be an odd integer. Then a $(2, s, 3)$-Cayley graph $X$ has a Hamilton cycle.

- $\langle x \rangle$ is corefree in $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle$:

  A method similar to the method used in $s \equiv 0 \pmod{4}$ case gives us a Hamilton cycle as a boundary of a Hamilton tree of faces consisting of hexagons and two $s$-gons.

- $\langle x \rangle$ is not corefree in $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots \rangle$:

  Results about lifts of Hamilton cycles in covers of graphs are needed.
Hamiltonicity of (2, s, 3)-Cayley graphs, s odd
Hamiltonicity of (2, s, t)-Cayley graphs

$X$ ... $(2, s, t)$-Cayley graph

$X_{2t}$ ... $2t$-gonal graph (arc-transitive graph admitting a 1-regular subgroup with cyclic vertex-stabilizer)

If the vertex set $V$ of $X_{2t}$ decomposes into $(I, V - I)$ with $I$ independent set and $V - I$ induces a tree then $X$ contains a Hamiltonian cycle.

Question: Characterize (tetravalent) arc-transitive graphs admitting a 1-regular subgroup via their decomposition properties.

In particular, characterize non-near-bipartite graphs amongst them.
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Thanks!