THE DISTINGUISHING INDEX OF 2-CONNECTED GRAPHS

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Graphs, groups, and more:
celebrating Brian Alspach's 80th
and Dragan Marušič's 65th birthdays

Koper, 29 May 2018

Distinguishing colouring

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Def. A colouring c of G is distinguishing if it breaks all non-trivial automorphisms of G.

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admissible graph is without more than one isolated vertex and without K_2 as a component.

The distinguishing index - examples

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- $D'(K_{1,m}) = m$

Upper bound for $D'(G)$

Thm. (K. & Pilśniak 2015)
If G is a connected graph of order $n \geq 3$, then

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unless $G \in \{C_3, C_4, C_5\}$.

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- a cycle C_n with $n \geq 6$,
- K_4 or $K_{3,3}$.

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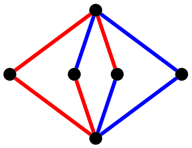

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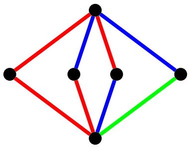

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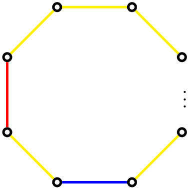
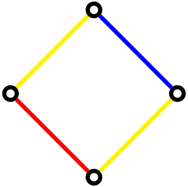


Outline of proof

- $c(G) = 4$



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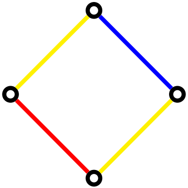

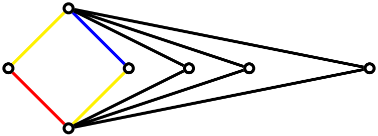

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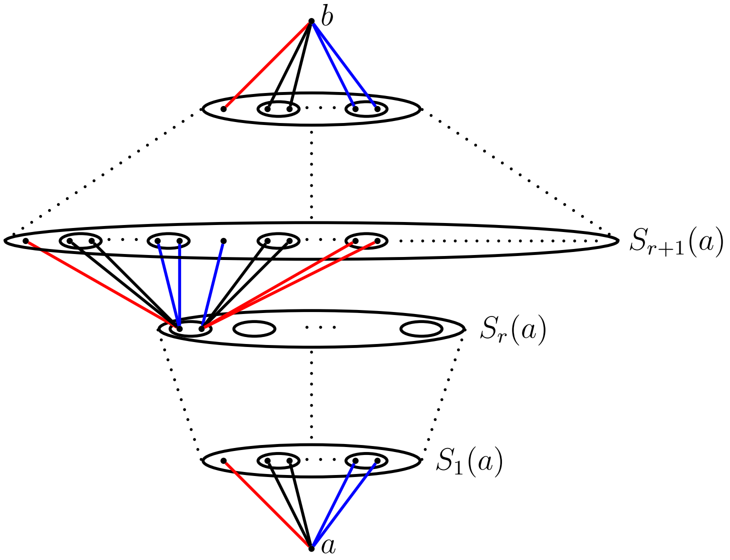

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We recursively colour the edges between $S_r(a)$ and $S_{r+1}(a)$ with $\left\lceil \sqrt{\Delta} \right\rceil$ colours such that for each r

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- $S_r(a)$ is fixed pointwise, whenever $S_{r+1}(a)$ is fixed so;
- if $A \subseteq S_{r+1}(a)$ is a set of vertices that can be interchanged, then $|A| \leq \left\lceil \sqrt{\Delta} \right\rceil$.

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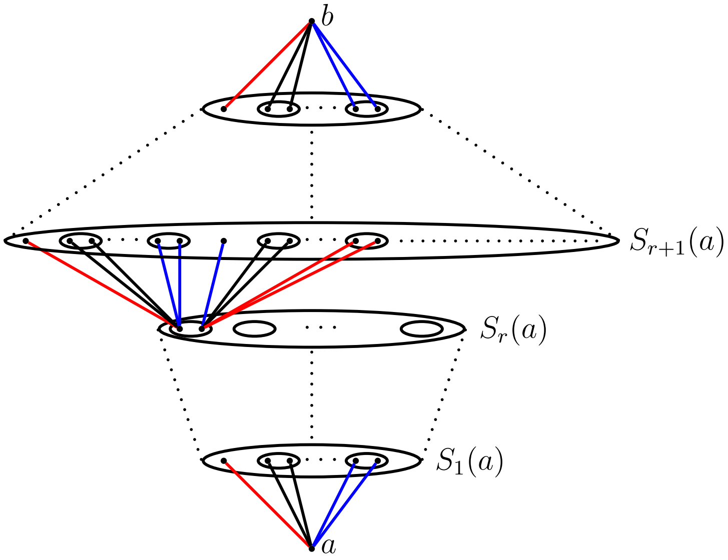

D' for $\delta \geq 2$

Thm. (IKPW 2018+)
If G is a connected graph with $\delta(G) \geq 2$, then

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If both G and $\overline{G}$ are admissible graphs of order $n \geq 7$, then

$$2 \leq D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Conjecture

Conj. (IKPW) If G is a connected graph of order at least 7 and $\delta(G) \geq 2$, then

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$|G| \geq 7$ since $D'(K_{3,3}) = 3$

THANK YOU !!