Graphs, groups, and more: celebrating Brian Alspach's 80th and Dragan Marušič's 65th birthdays

Configurations of points and conics

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Definition

By a configuration of points and conics of type (p_q, n_k) we mean a set consisting of p points and n conics such that each point is incident with precisely q conics, and precisely k points are sitting on each conic.

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We consider these configurations as embedded in the real projective plane.

Definition

The Levi graph of a (p_q, n_k) configuration $\mathcal{C}$ is a bipartite (q, k)-regular graph $L(\mathcal{C})$ whose set of vertices is in bijective correspondence with the elements of $\mathcal{C}$ such that two vertices in $L(\mathcal{C})$ are adjacent if and only if the corresponding elements in $\mathcal{C}$ are incident.

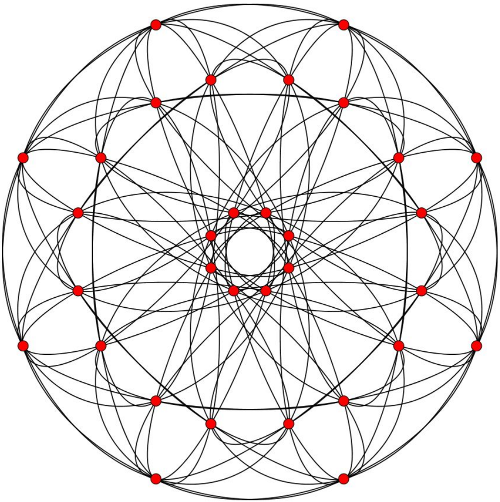
Example: PE-(32_6)

(GG 2009)

It is based on the regular map of type $\{6,6\,|\,3,4\}$ introduced by Coxeter (1937) (= the map C17.3 in Marston Conder's list).

The ellipses are circumscribed around the hexagonal faces of the map.

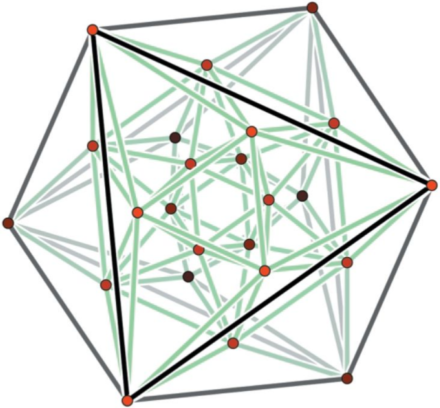
The map can be represented within a 4-dimensional cube; hence the configuration can be derived directly from this cube, too.



An analogous case: PE-(96_6)

(96 points, 96 ellipses)

It is constructed from the regular 24-cell (besides the 4-cube, another 4-dimensional regular polytope).



Constructions (from maps)

A generalization of the PE-(32_6) example:

Doubly infinite family of point-conic configurations PCo-$((2mn)_6)$

- type: $((2mn)_6)$ (for even numbers $m, n \geq 4$);
- start from the Cartesian product of a regular m-gon and a regular n-gon (a 4-dimensional convex polytope called prismotope; it has n copies of m-sided prisms and m copies of n-sided prisms as facets);
- inscribe (mirror-symmetric) hexagons in these prismatic facets (generically, these hexagons form an equivelar $\{6, 6\}$ map);
- circumscribe ellipses around these hexagons;
- project the system of vertices and ellipses onto a suitable plane.

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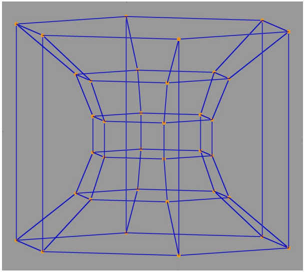
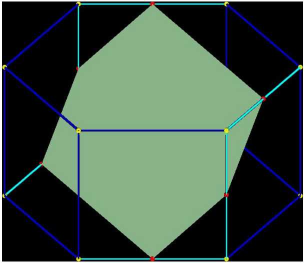
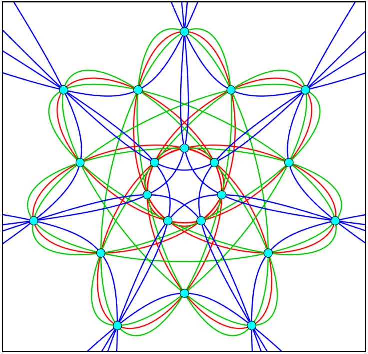


The Cartesian product of two hexagons

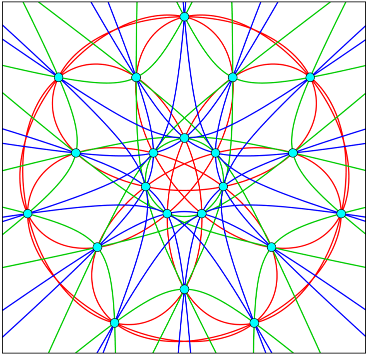


Hexagon inscribed in a prismatic facet

Two examples of type (21_7)

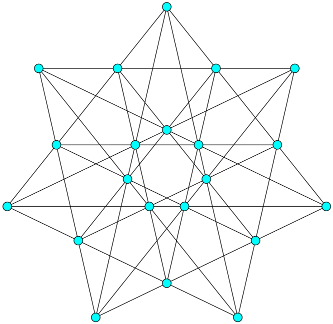


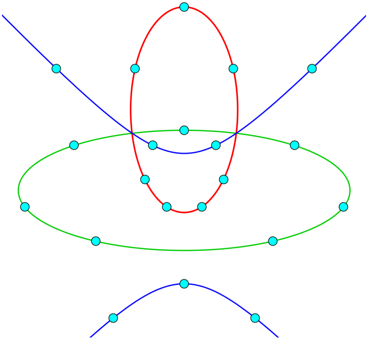
EEH-(21_7)



EHH-(21_7)

Two examples of type (21_7) (constructed from KGR-(21_4))

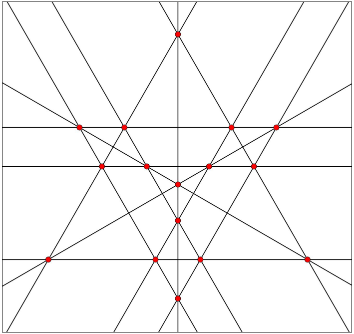




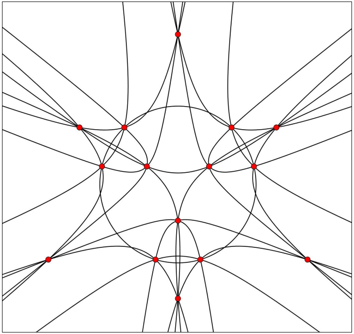
An observation by Luis Montejano
and personal communication of Leah Berman.
Proof of existence: GG (2018)

A non-balanced example of type $(15_4, 10_6)$

Derived from a point-line Z_1-$(16_3, 12_4)$. (Movable!)



Z_1-$(16_3, 12_4)$ (Zacharias, 1941)

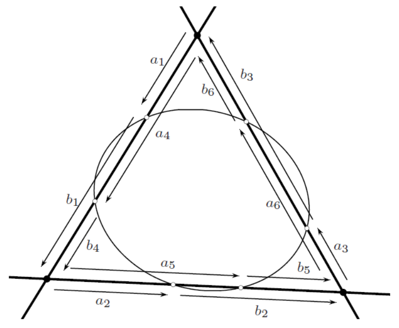


PCo-$(15_4, 10_6)$

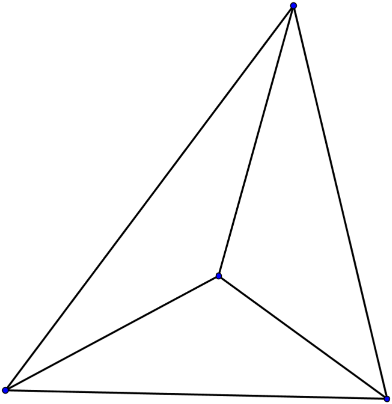
THEOREM (Lazare Carnot 1806)

Consider a triangle with exactly two (distinct) edge points per edge. We assume that the edge points are labeled $1, \ldots, 6$ and that the corresponding length ratios are a_i/b_i. Then the six points lie on a common conic if and only if the following relation holds:

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} \cdot \frac{a_4}{b_4} \cdot \frac{a_5}{b_5} \cdot \frac{a_6}{b_6} = 1.$$



Constructions (based on incidence theorems)



Jürgen Richter-Gebert (2011)

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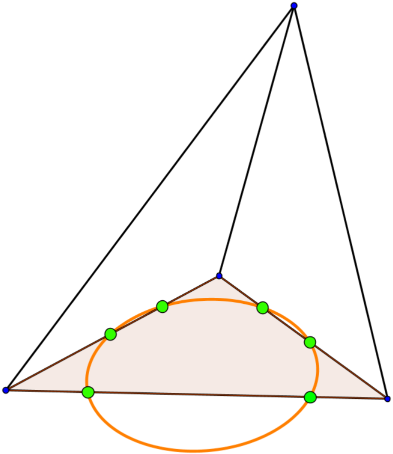

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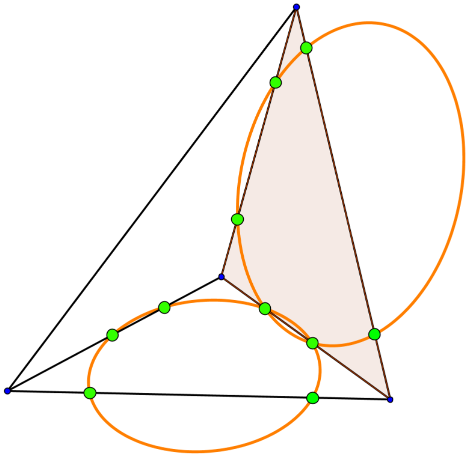

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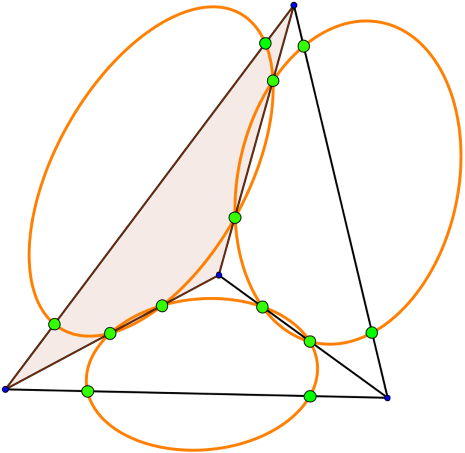

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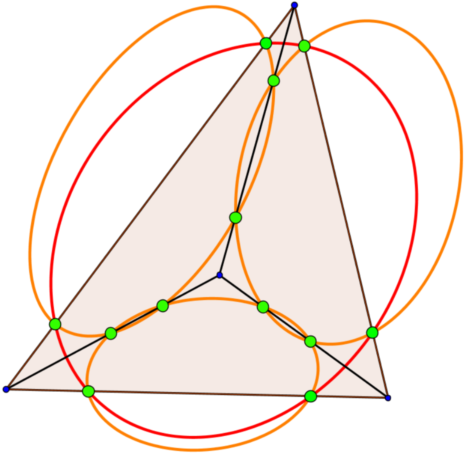

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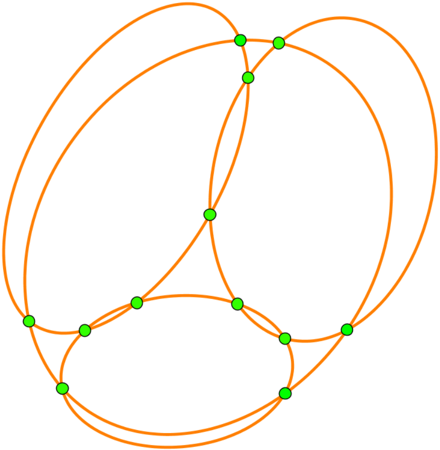

PCo-$(12_2, 4_6)$

Constructions (based on incidence theorems)

THEOREM (Richter-Gebert 2011)

The incidence theorem for the tetrahedron above can be extended to an arbitrary triangulated compact 2-manifold: if the sixtuples of points for all but one triangle are coconical, then the last sixtuple is coconical as well.

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Applications:

- An infinite sequence of type $\left((6n)_2, (2n)_6\right)$:
  for each integer $n \geq 3$, there is a point-conic configuration of type $\left((6n)_2, (2n)_6\right)$ derived from an n-gonal dipyramid. Non-balanced...

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- A balanced example derived from 7-dimensional simplex: PCo-(56_6).

Cartesian product of configurations

Definition (Pisanski and Servatius 2013, GG 2014)

Let $\mathcal{C}$ be configuration of type (v_r, b_k) and $\mathcal{C}'$ a configuration of type $(v'_{r'}, b'_k)$. Observe that these two configurations have the same number k of points in each block. The Cartesian product of $\mathcal{C}$ and $\mathcal{C}'$ is a configuration of type

$$\left((vv')_{(r+r')}, (vb' + v'b)_k\right),$$

whose point set is the Cartesian product of the point sets of $\mathcal{C}$ and $\mathcal{C}'$ and where there is a block incident to two points (x, x') and (y, y') if and only if either $x = y$ and there is a block incident to x' and y' in $\mathcal{C}'$, or $x' = y'$ and there is a block incident to x and y in $\mathcal{C}$.

Constructions

Cartesian product of configurations

Non-balanced $\longrightarrow$ balanced configuration

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The point set of the product configuration is the Minkowski sum of the point sets of the component configurations:

$$P = \{x + y \,|\, x \in P_1, y \in P_2\}.$$

Definition

A configuration $\mathcal{C}$ is called

- lineal if any two blocks are incident with at most one point;
- circular if any two blocks are incident with at most two points;
- conical if any two blocks are incident with at most four points.

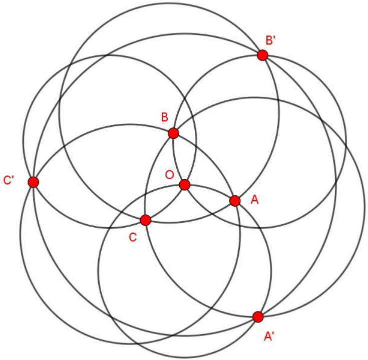
In terms of Levi graphs:

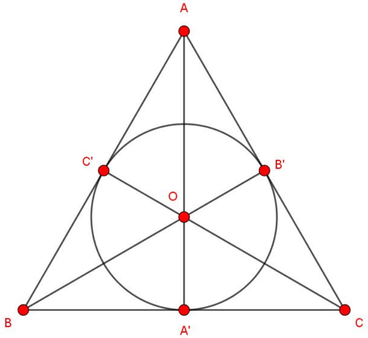
$$\mathcal{C} \text{ is } \left\{ \begin{array}{l} \text{lineal} \\ \text{circular} \\ \text{conical} \end{array} \right\} \text{ if } L(\mathcal{C}) \text{ contains no } \left\{ \begin{array}{l} K_{2,2} \\ K_{3,2} \\ K_{5,2} \end{array} \right\} \text{ subgraph.}$$

Realization problems

Not every lineal configuration can be realized by points and lines!

Example: the (7_3) Fano configuration





It can be realized by circles (trivial), but not with lines.

QUESTION.
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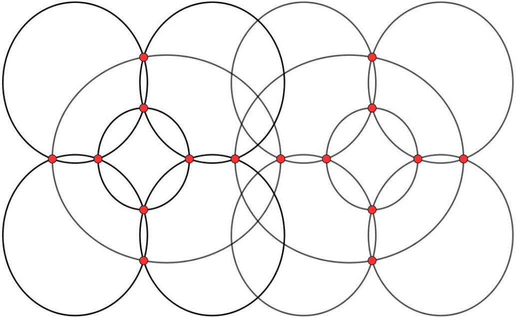

PE-$(16_3, 12_4)$

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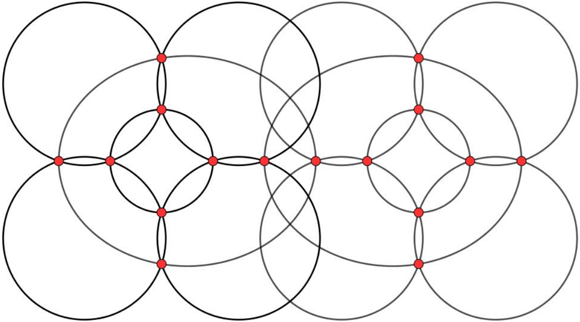
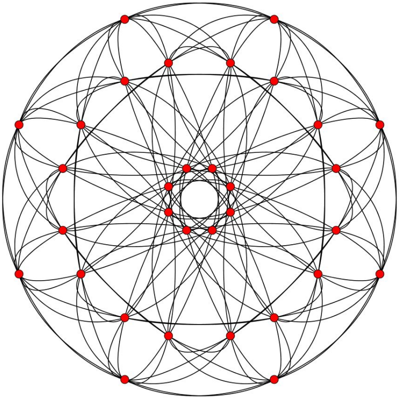


PE-$(16_3, 12_4)$

PROBLEM.

Can this configuration be realized by circles?

Open...



PE-(32_6)

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Is there a point-conic configuration which can only be realized by ellipses? (**Open...**)

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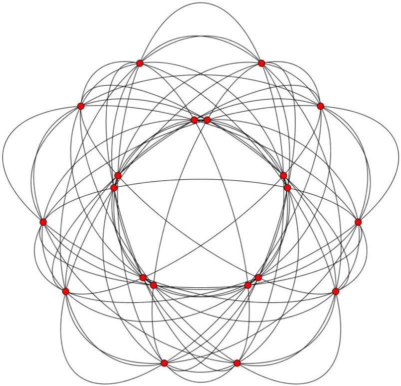
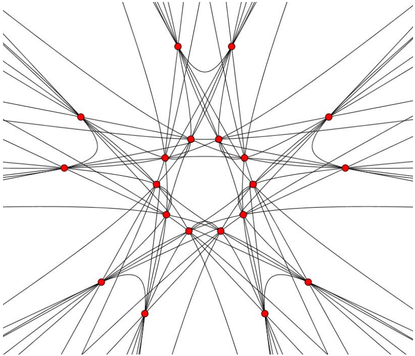
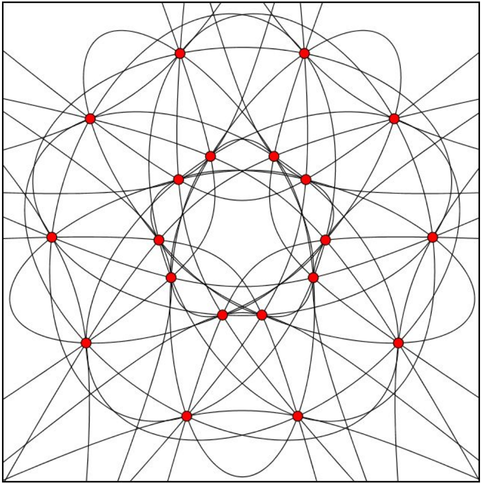




PCo_1-(20_6)

End



End

Thank you for
your attention.