Odd Cycle Bases of Nonbipartite Graphs

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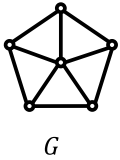
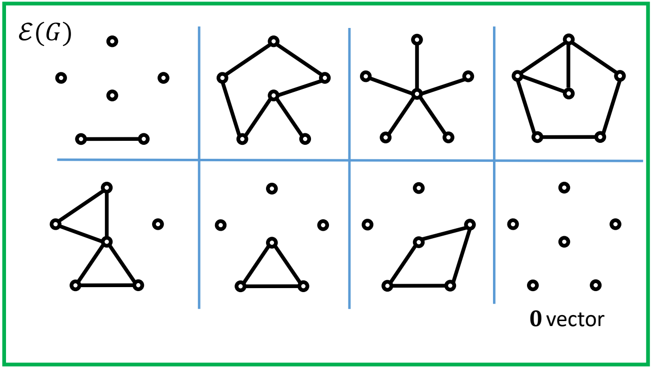
Edge Space of a Graph

Definition

edge space of G: elements are spanning subgraphs of G
operation is the symmetric difference of edge sets

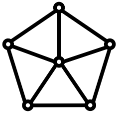




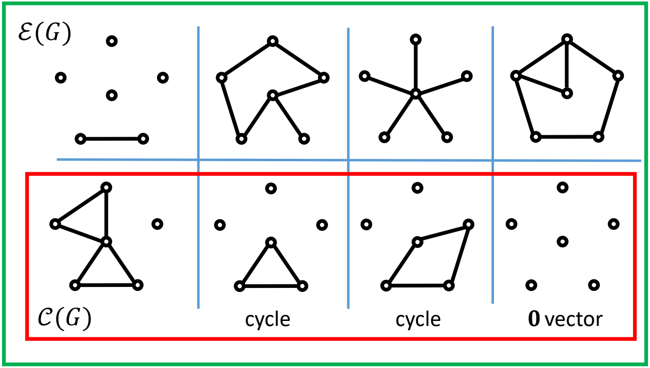
Cycle Space of a Graph

Definition

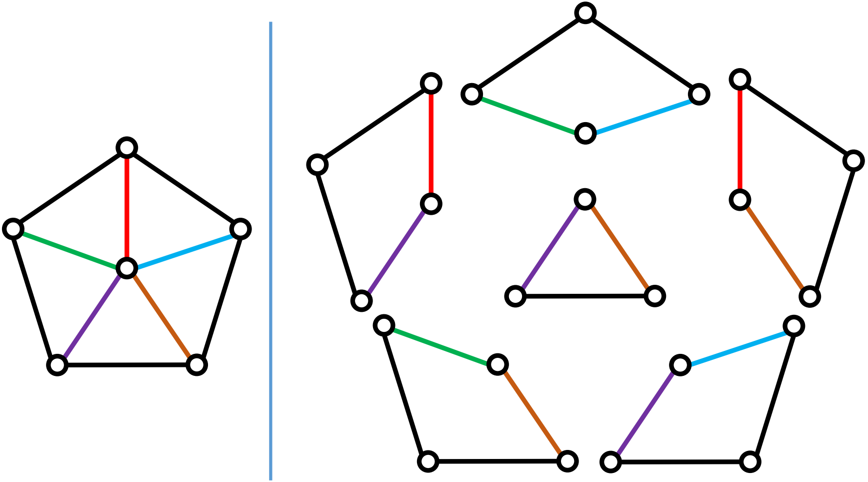
cycle space: subspace of edge space spanned by the graph's cycles
(equivalently: all even spanning subgraphs)



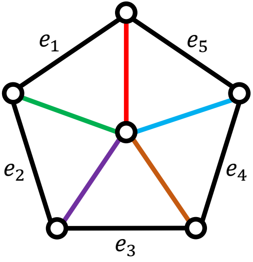
G

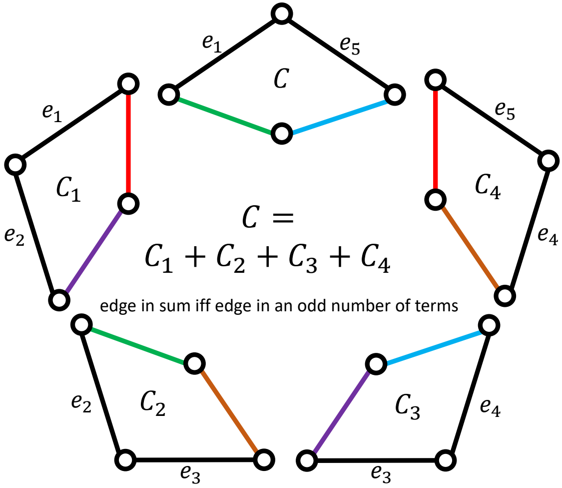


Cycles that Generate the Cycle Space of the Wheel

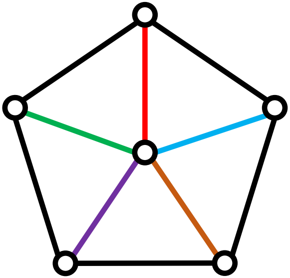


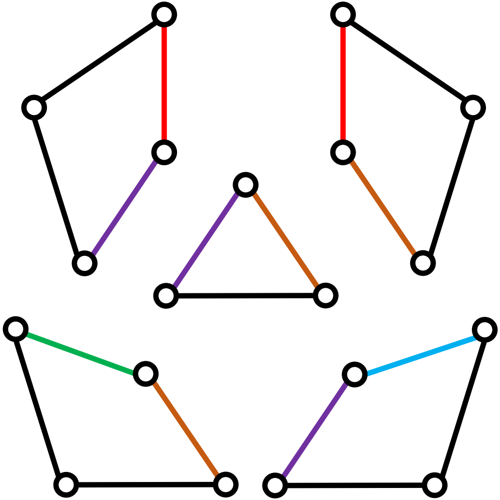
Not a Basis of the Cycle Space of the Wheel



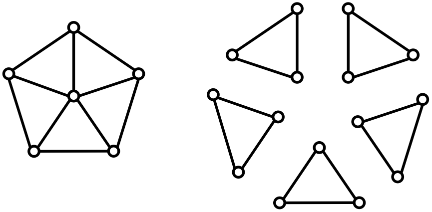


A Basis of the Cycle Space of the Wheel





Another Cycle Basis of the Wheel



simple and odd cycle basis

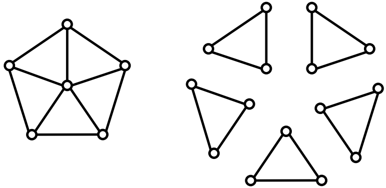
Definition

A cycle basis $\mathcal{B}$ of a graph G is simple if every edge of G is in at most two cycles of $\mathcal{B}$.

Definition

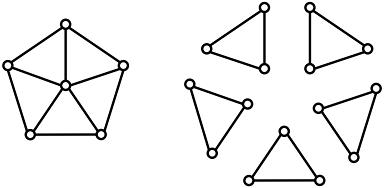
A cycle basis $\mathcal{B}$ of a graph G is odd if every cycle of $\mathcal{B}$ is of odd length.

Another Cycle Basis of the Wheel



Why do we care about cycle bases?

Another Cycle Basis of the Wheel



Why do we care about cycle bases?

- applications:
  - Kirchhoff's Voltage Law
  - periodic scheduling in traffic planning
  - graph drawing

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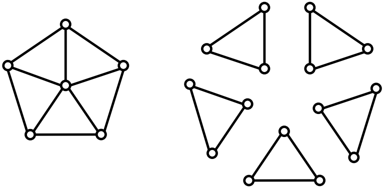


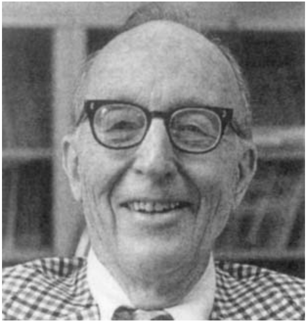
Why do we care about cycle bases?

- applications:
  - Kirchhoff's Voltage Law
  - periodic scheduling in traffic planning
  - graph drawing
- theorems regarding the structure of graphs:
  - algebraic characterization of planarity
    (simple basis above proves the wheel is planar!)
  - counting the number odd cycles in nonbipartite graphs

Algebraic Characterization of Planarity

Theorem (Saunders MacLane 1937)

A graph is planar if and only if its cycle space has a simple basis.



1989 in Dunedin, New Zealand
(Saunders was 80 then (too))

Brian's Influence On Me

- cycles, cycles, cycles and more cycles
- Brian and I published one paper together:
  "Edge-Pancyclic Block-Intersection Graphs" 1991
  - by far my most cited (39)
  - only his 8th most cited
  - fault-tolerant networkers started to use the result in 2005

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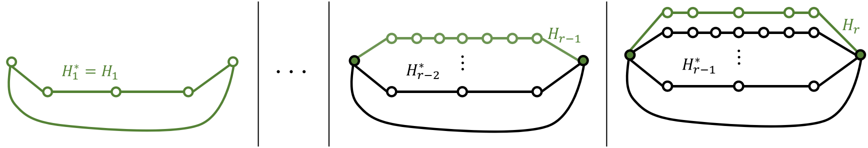
  - fault-tolerant networkers started to use the result in 2005
- during my PhD (1987-1991), Brian secured a provincial scholarship where I worked with a company to develop a staff scheduling application
- a few years later, with a math position in Kelowna, I obtained funding for a research project for a local staff scheduling company
- this led to a discrete optimization consulting career using CPLEX and constraint programming tools
- I got my position reduced to half time 15 years ago, but in a month from now I go back full time

Ear Decompositions

Whitney's theorem (1932) states that every 2-connected graph G has
an ear decomposition $(H_1, \ldots, H_r)$ where:

1. H_1 is any cycle of G,
2. defining $H_1^* = H_1$ and for $i = 1, \ldots, r-1$,
   $H_{i+1}^* = H_i^* \cup H_{i+1}$,
   each H_{i+1} is a H_i^*-path in G,
3. $H_r^* = G$, and
4. necessarily $r = r(G) = m - n + 1$.

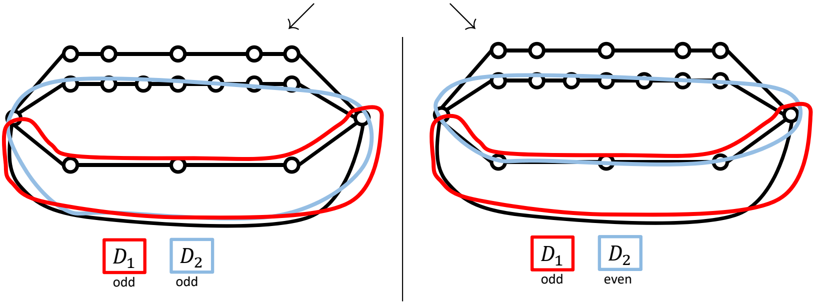


the path H_{i+1} is called an ear of H_i^* in G

Cycle Bases From Ear Decompositions

- each ear can be extended to a cycle and the collection of these cycles is a basis for the cycle space of the graph
- if an odd cycle is chosen initially, then each ear can be extended to a cycle of either parity (Henning and Little 1999, H. 2017+)

either



2-Connected Nonbipartite Graphs Have Cycle Bases With Prescribed Parities

Odd Cycles in 2-Connected Nonbipartite Graphs

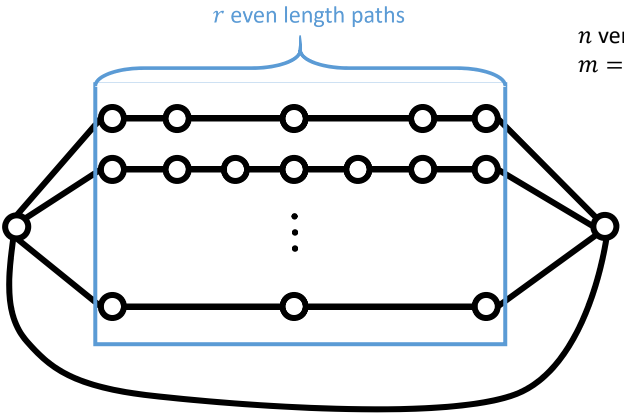
Theorem (H. 2017+)

The cycle space of a 2-connected nonbipartite graph is generated by its odd cycles.

Corollary

 # odd cycles of a 2-connected nonbipartite G is at least $r(G) = m - n + 1$ and the limiting example shows that this is best possible

The Limiting Example



n vertices
$m = n + r - 1$ edges

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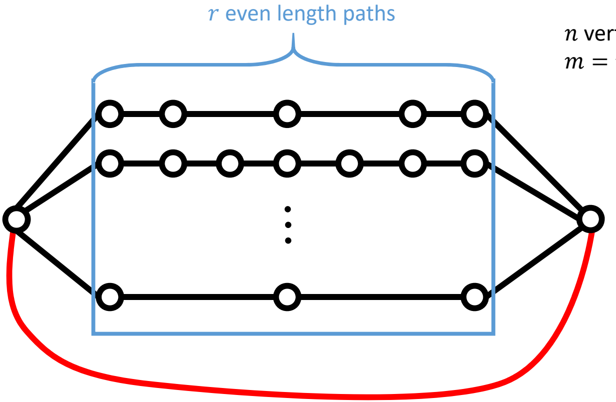

n vertices
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all odd cycles use
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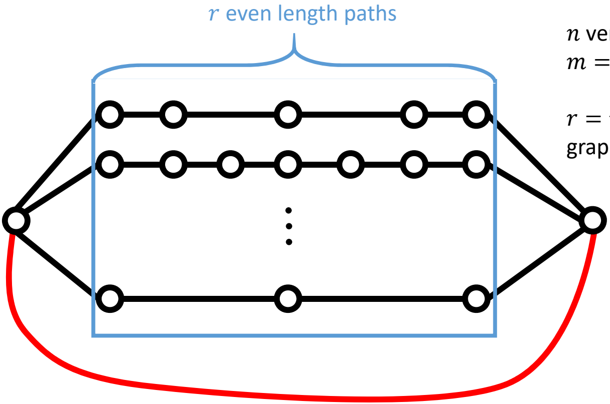


n vertices
$m = n + r - 1$ edges

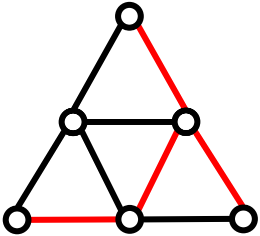
$r = m - n + 1$ is the graph's cyclomatic number

all odd cycles use red edge so there are exactly r odd cycles

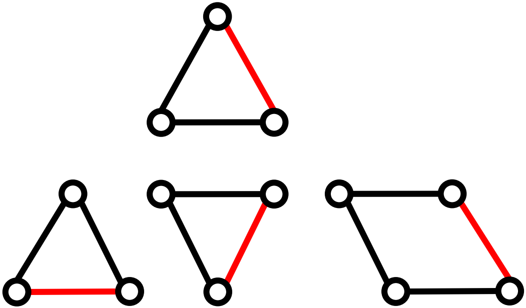
thus: upper bound on the minimum number of odd cycles in a 2-connected nonbipartite graph is the graph's cyclomatic number

Another Way to Form a Cycle Basis

- choose a spanning tree $T \subseteq G$ (black edges)
- every non-tree edge e of G (red edges) creates a unique cycle in $T + e$ for a cycle basis of G



the sunflower graph $SF(3)$

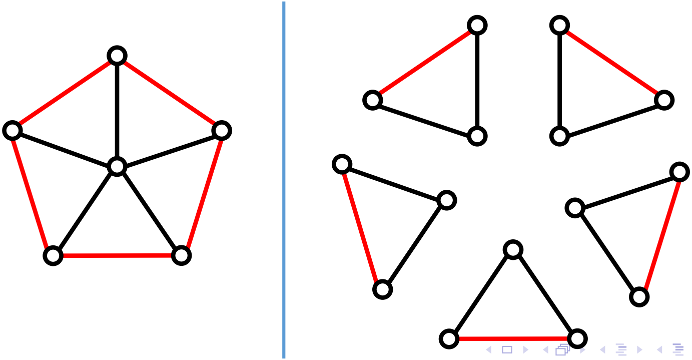


a cycle basis from the spanning tree

Fundamental Cycle Basis

Definition

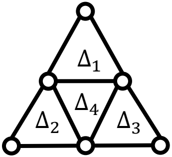
a cycle basis of a graph G is fundamental if there is exists a spanning tree $T \subseteq G$ (black edges) such that for every non-tree edge e of G (red edges), the unique cycle in $T + e$ is a member of the basis

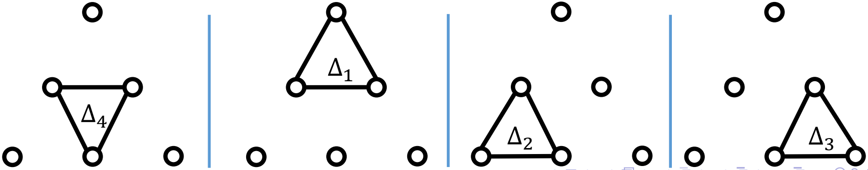


Is This Odd Cycle Basis Fundamental?

Definition

a cycle basis of a graph G is fundamental if there is exists a spanning tree $T \subseteq G$ such that for every non-tree edge e of G, the unique cycle in $T + e$ is a member of the basis





Not Every Cycle Basis is Fundamental

Definition

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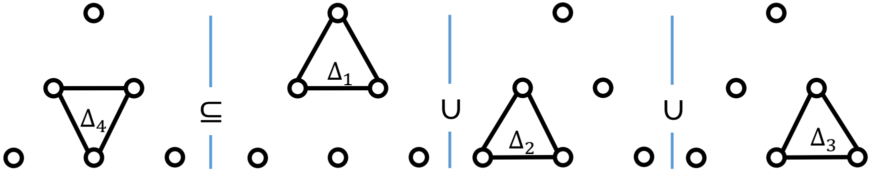
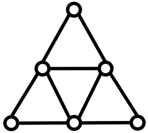


Theorem (Hubicka and Sysło 1975)

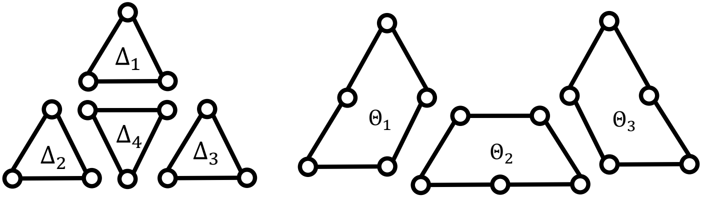
A cycle basis $\mathcal{B}$ of G is fundamental if and only if every cycle of $\mathcal{B}$ has an edge that is in no other cycle of $\mathcal{B}$.

Not Every Graph has a Fundamental Odd Cycle Basis

$SF(3)$:



all odd cycles:



- every cycle basis $\mathcal{B}$ has $|\mathcal{B}| = 4 = r(SF(3))$
- the 3-cycles form an odd cycle basis

Not Every Graph has a Fundamental Odd Cycle Basis

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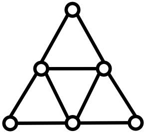

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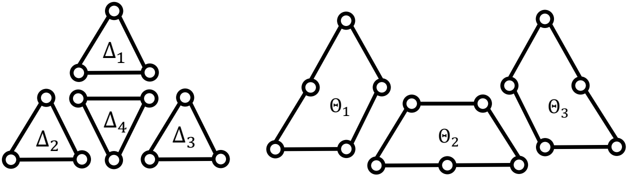

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- suppose $\mathcal{B}$ is a fundamental odd cycle basis

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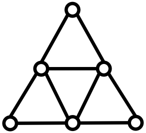

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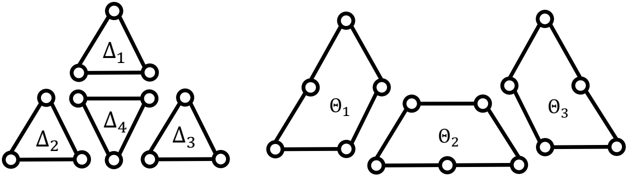

- every cycle basis $\mathcal{B}$ has $|\mathcal{B}| = 4 = r(SF(3))$
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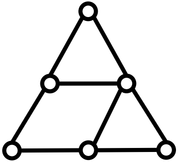
- union of two 5-cycles covers all but one edge of $SF(3)$



$\Theta_1 \cup \Theta_2$

Not Every Graph has a Fundamental Odd Cycle Basis

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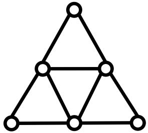

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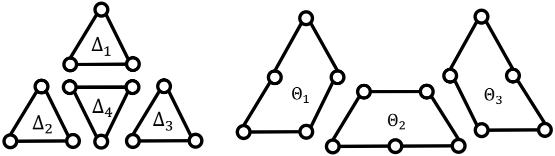

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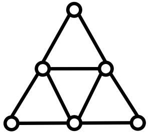

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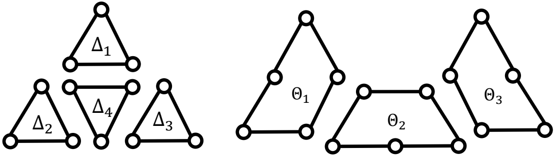

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- $\Longrightarrow \mathcal{B}$ contains at most one 5-cycle
- $\Delta_1 \cup \Delta_2 \cup \Delta_3 = SF(3) \Longrightarrow \{\Delta_1, \Delta_2, \Delta_3\} \not\subseteq \mathcal{B}$

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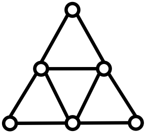

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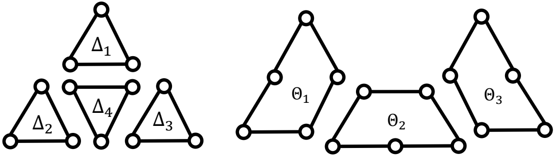

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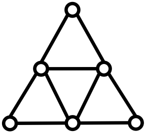

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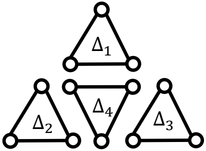

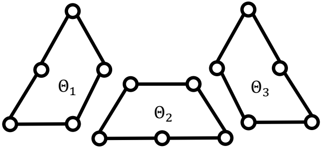

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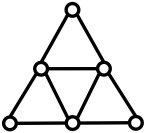

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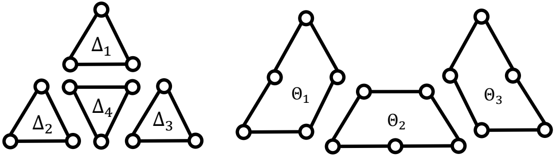

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Not Every Graph has a Fundamental Odd Cycle Basis

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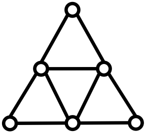

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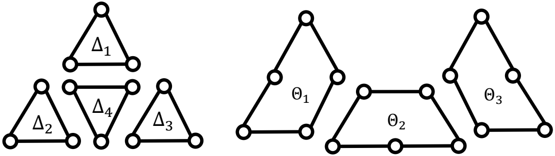

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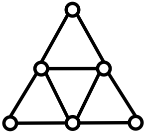
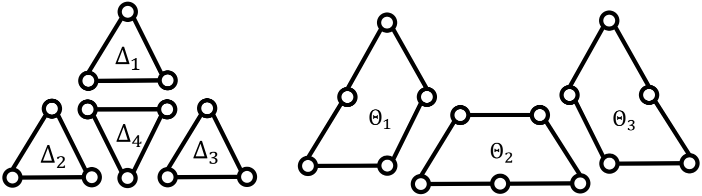
- but $\Delta_2 \subset \Delta_4 \cup \Theta_2 \Rightarrow\Leftarrow$

Not Every Graph has a Fundamental Odd Cycle Basis

$SF(3)$:



all odd cycles:



- conclusion: $SF(3)$ has no fundamental odd cycle basis
- moral: can't necessarily find an odd cycle basis using a spanning tree but can with an ear decomposition

3–Connected Nonbipartite Graphs

Theorem (Bondy and Lovász 1981)

Let R be a set of $s-1$ vertices in an s-connected graph G. Then the cycles through R generate the cycle space of G.

- can't specify $s-1$ vertices if cycles must be odd

3–Connected Nonbipartite Graphs

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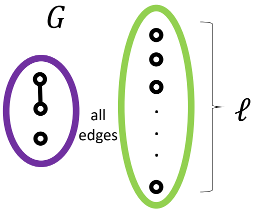
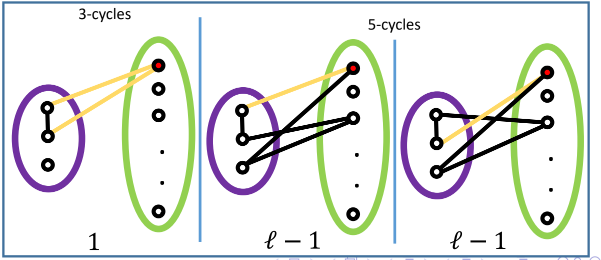
Let R be a set of $s-1$ vertices in an s-connected graph G. Then the cycles through R generate the cycle space of G.

- can't specify $s-1$ vertices if cycles must be odd
- counterexample by Toft (1975) used for this purpose ($s=3$, $\ell \geq 3$)



$r(G) = 2\ell - 1$



3- vs 2-Connected Nonbipartite Graphs

Theorem (H. 2017+)

The cycle space of a 3-connected nonbipartite graph is generated by the odd cycles through any fixed vertex of the graph.

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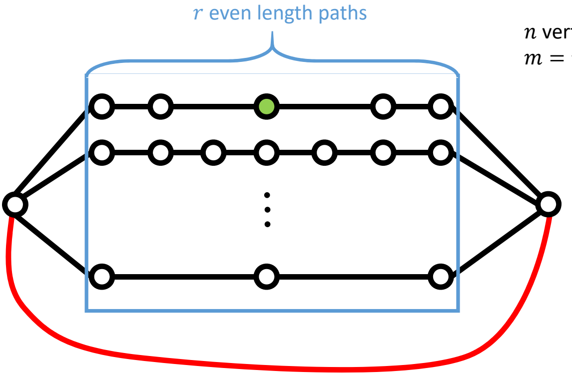
not the case for 2-connected nonbipartite graphs:



n vertices
$m = n + r - 1$ edges

all odd cycles use red edge so there is exactly 1 odd cycle through ●

Theorem (H. 2017+)

The cycle space of a 3-connected nonbipartite graph G is generated by the odd cycles through any fixed vertex of the graph.

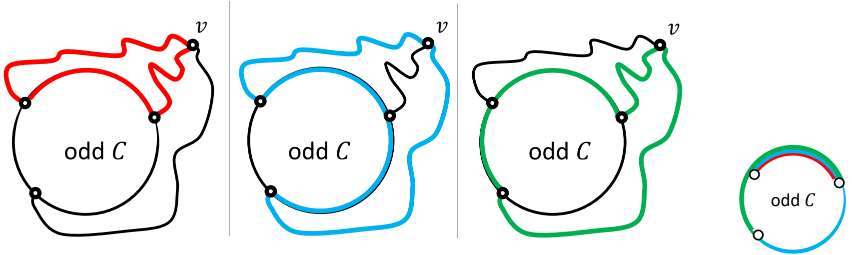
Proof

Let $v \in V(G)$.

- may assume $G - v$ is nonbipartite since otherwise all odd cycles contain v
- $G - v$ is 2-connected and so has an odd cycle basis $\mathcal{B}$
- consider $C \in \mathcal{B}$

Lemma

If G is a 3-connected graph containing an odd cycle C and v is any vertex of G not on C, then v is on three distinct odd cycles C_1, C_2, C_3 of G such that $C = C_1 + C_2 + C_3$.



odd C_1 odd C_2 $\Longrightarrow$ odd C_3 and $C = C_1 + C_2 + C_3$

Theorem (H. 2017+)

The cycle space of a 3-connected nonbipartite graph G is generated by the odd cycles through any fixed vertex of the graph.

Proof

Let $v \in V(G)$.

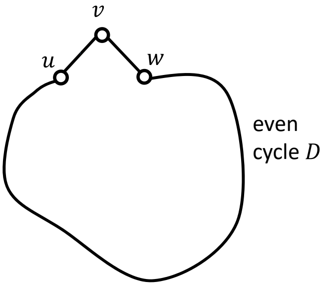
- may assume $G - v$ is nonbipartite since otherwise all odd cycles contain v
- $G - v$ is 2-connected and so has an odd cycle basis $\mathcal{B}$
- consider $C \in \mathcal{B}$
  - C is generated by odd cycles through v
  - thus all cycles of $G - v$ are generated by odd cycles through v

Theorem (H. 2017+)

The cycle space of a 3-connected nonbipartite graph is generated by the odd cycles through any fixed vertex of the graph.

Proof continued...

- since odd cycles through v generate the odd cycles through v, final case:
- let D be an even cycle of G that contains v

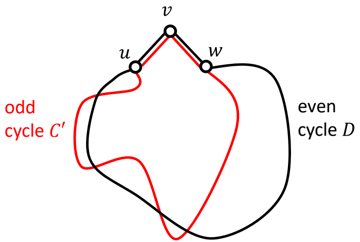


Theorem (H. 2017+)

The cycle space of a 3-connected nonbipartite graph is generated by the odd cycles through any fixed vertex of the graph.

Proof continued...

- since $G - v$ is nonbipartite, by McCuaig and Rosenfeld(1985), there is an odd cycle C' in G containing both uv and vw

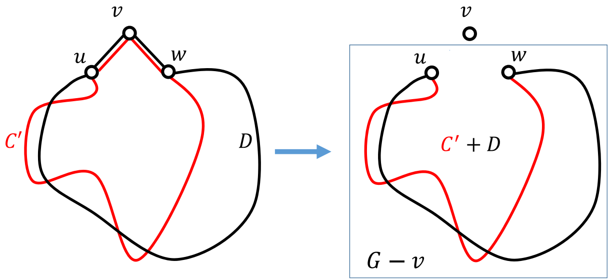


Theorem (H. 2017+)

The cycle space of a 3-connected nonbipartite graph is generated by the odd cycles through any fixed vertex of the graph.

Proof continued...

- thus the edge set of the even subgraph $C' + D$ is contained in $E(G - v)$



Theorem (H. 2017+)

The cycle space of a 3-connected nonbipartite graph is generated by the odd cycles through any fixed vertex of the graph.

Proof continued...

- thus the edge set of the even subgraph $C' + D$ is contained in $E(G - v)$
- therefore there are $C_1, \ldots, C_k \in \mathcal{B}$ such that

$$\begin{aligned} C' + D &= C_1 + \cdots + C_k \\ \Rightarrow D &= C' + C_1 + \cdots + C_k \end{aligned}$$

- odd cycles through v generate C' and $C_1, \ldots, C_k$, and so generate D $\square$

Theorem (H. 2017+)

The cycle space of a 3-connected nonbipartite graph is generated by the odd cycles through any fixed vertex of the graph.

open question: for 4-connected nonbipartite graphs, do the odd cycles through any pair of fixed vertices generate the graph's cycle space?

open question: what is the minimum number of odd cycles in a 3-connected nonbipartite graph? (H. 2017+: $\geq 2(r-1) \geq n$) ... should be $> \frac{(n-1)(n-2)}{2}$

Thanks to the organizers of the conference
and to Brian (and of course Kathy)