Graphs, groups, and more

Koper, May 28-June 1, 2018

Maurizio Brunetti

University of Naples "Federico II"

Edge perturbations on signed graphs with clusters

(Joint research with Francesco Belardo & Adriana Ciampella)

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A Tale of Two (mathematical) Cities

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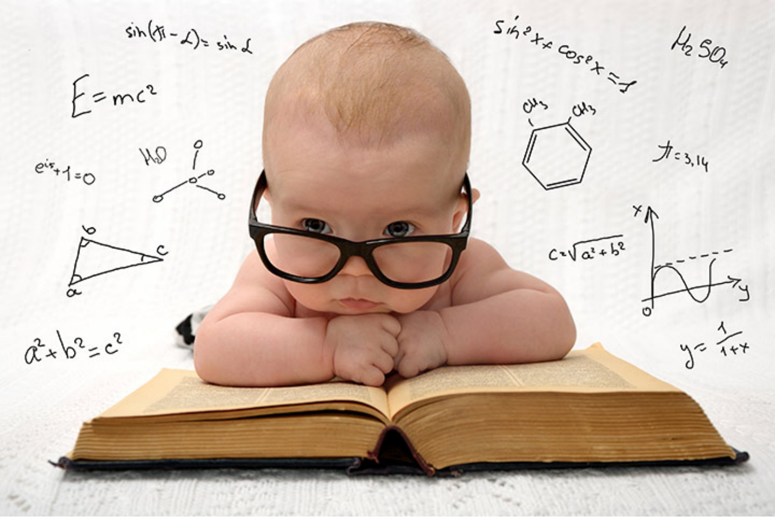

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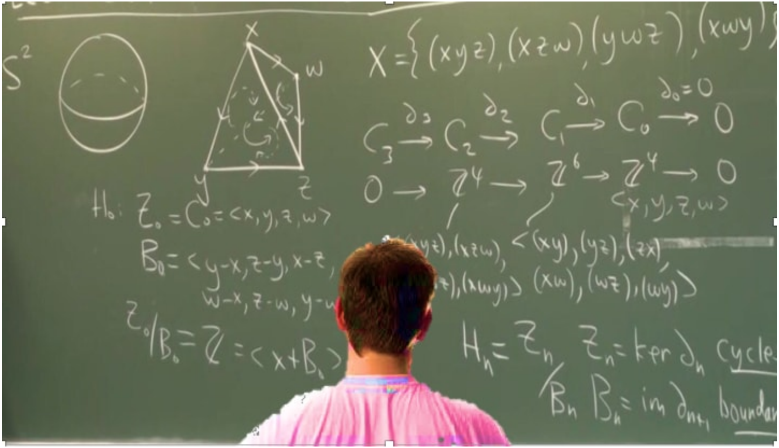
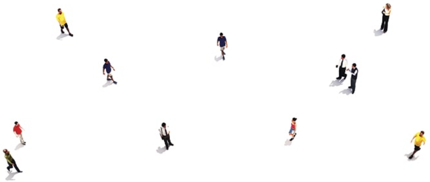


The first city: Algebraic Topology



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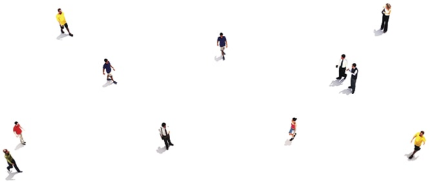

⋆ Generalized Cohomology Theories (55N20)

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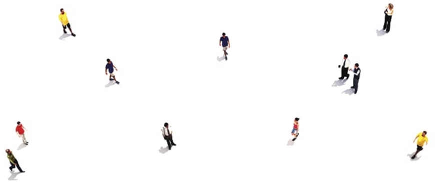

- ⋆ Generalized Cohomology Theories (55N20)
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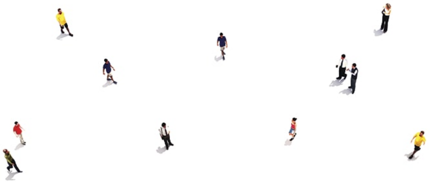

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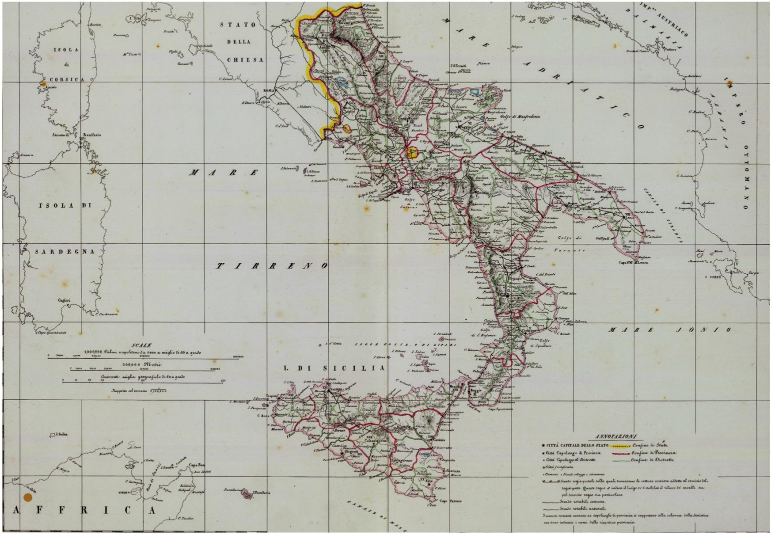
- ⋆ Steenrod Algebra & Cohomology Operations (55S10)

Topography


STATO DELLA CHIESA
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The second city: Graph Theory

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The irresistible beauty of (signed) graphs and their spectra

Graphs, groups, and more

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Thank you!

(End of biographical sub-talk)

Signed Graphs

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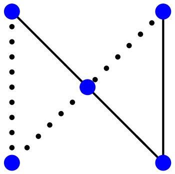
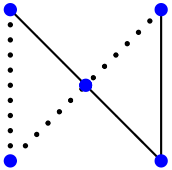


Example of a signed graph.

positive edges = solid lines;
negative edges = dotted lines.

More on Signed Graphs

If C is a cycle in Γ, the sign of C, denoted by $\sigma(C)$, is the product of its edges signs.



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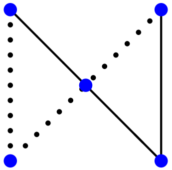


An unbalanced butterfly

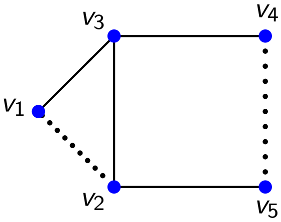
Definition

A signed graph is said to be balanced if and only if all its cycles are positive.

Adjacency matrix of Signed Graphs

The adjacency matrix is defined as $A(\Gamma) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i v_j), & \text{if } v_i \sim v_j; \\ 0, & \text{if } v_i \not\sim v_j. \end{cases}$$



$$\rightsquigarrow \quad A(\Gamma) = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

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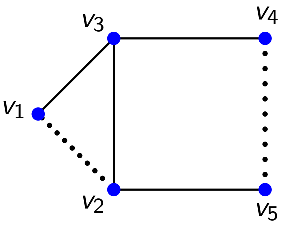

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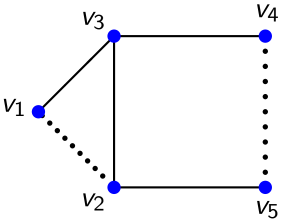

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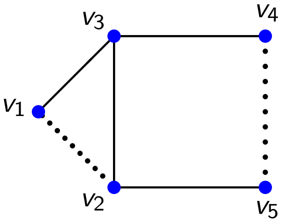

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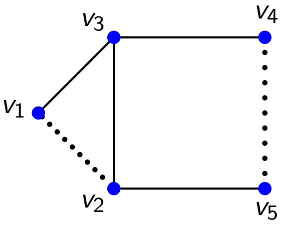
Laplacian of Signed Graphs

The Laplacian matrix of $\Gamma = (G, \sigma)$ is defined as
$L(\Gamma) = D(G) - A(\Gamma) = (l_{ij})$

$$l_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j; \\ -\sigma(v_i v_j), & \text{if } i \neq j. \end{cases}$$



$$\rightsquigarrow \quad L(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 2 \end{pmatrix}$$

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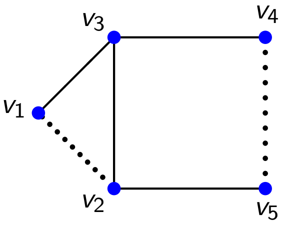

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Adjacency and Laplacian eigenvalues

Proposition

Adjacency eigenvectors are all real. Laplacian eigenvectors are all real and non-negative

$$\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma) \quad \text{Adjacency eigenvalues}$$

$$\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) \geq 0 \quad \text{Laplacian eigenvalues}$$

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D.M. Cardoso, D. Cvetković, P. Rowlinson, S.K. Simić, *A sharp lower bound for the least eigenvalue of the signless Laplacian of a non-bipartite graph*, Linear Algebra Appl. 429 (2008) 2770–2780.

F. Belardo, *Balancedness and the least eigenvalue of Laplacian of Signed Graphs*, Linear Algebra Appl. 446 (2014) 133–147.

The least eigenvalue

A celebrated result by T. Zaslavsky:

Theorem

Let $\Gamma = (G, \sigma)$ be a connected signed graph and $\mu_n(\Gamma)$ be its least Laplacian eigenvalue.
Then Γ is balanced if and only if $\mu_n(\Gamma) = 0$.

Graphs with clusters

What is a cluster?

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Definition

Let G a simple undirected graph. A (c, s)-cluster is a couple of vertex subsets (C, S) with the following property. The $c \geq 2$ vertices of C all have the same set S of neighbors. The set S has cardinality s.

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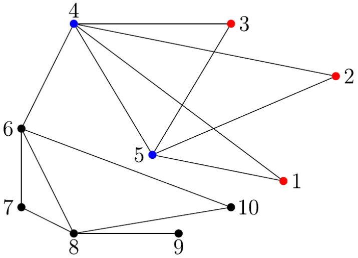
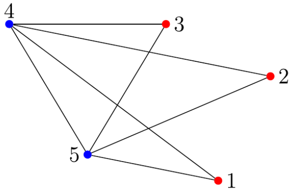


G



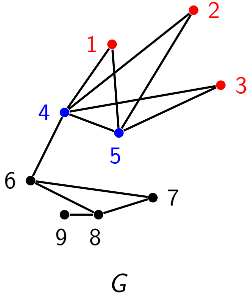
cluster (C, S)

$C = \{1, 2, 3\}$ and $S = \{4, 5\}$

Edge perturbation = Adding edges

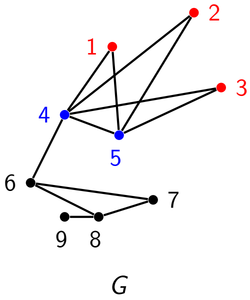
Given a graph G with a (c,s)-cluster (C,S)

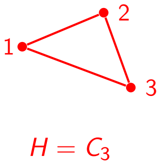
,



Edge perturbation = Adding edges

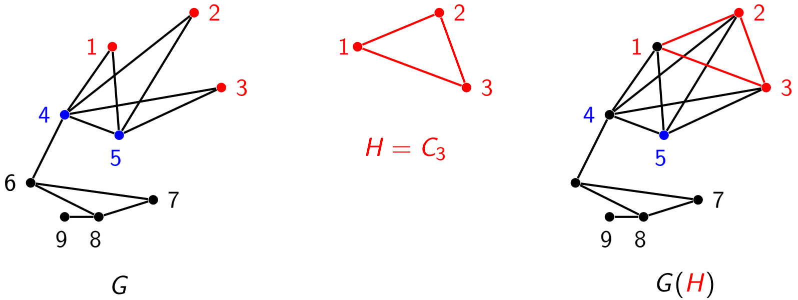
Given a graph G with a (c, s)-cluster (C, S) and any graph H of order c,





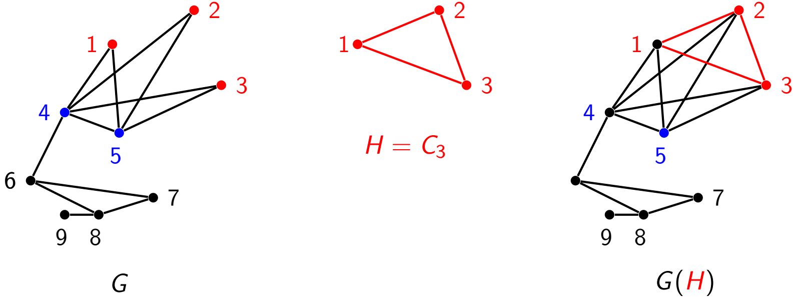
Edge perturbation = Adding edges

Given a graph G with a (c,s)-cluster (C,S)
and any graph H of order c, we can build $G(H)$.



Graphs with clusters

Is there any predictable relation among the A-spectra, the L-spectra, and the Q-spectra of G, H, and $G(H)$?



Graphs with clusters

A Theorem by D. Cardoso & O. Rojo (2017):

Let M be the adjacency matrix A, the Laplacian matrix L or the signless Laplacian matrix Q.

according that $M(H)\mathbf{1}_c = \mu_c(M(H))\mathbf{1}_c$.

$$\det(\lambda I - M(G(H))) = g_M^H \prod_{i=1}^{c-1} (\lambda - (s\delta + \mu_i(M(H)))).$$

where

$$g_M^H = p_{M(G)}(\lambda) - (\mu_c(M(H)) + s\delta)q_{M(G)}(\lambda).$$

Main Condition/Restriction

$$M(H)\mathbf{1}_c = \mu_c(M(H))\mathbf{1}_c.$$

- $\mathbf{1}_c$ is an L-eigenvector for all graphs H (of order c)
- $\mathbf{1}_c$ is an A-eigenvector and a Q-eigenvector for the graph H if and only if H is k-regular.

Signed Regularities

Let $\Lambda = (H, \tau)$ be a signed graph.

When $\quad M(\Lambda)\mathbf{1}_c = \mu_c(M(\Lambda))\mathbf{1}_c \quad ?$

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For M being the adjacency matrix, this happens when Λ is *net-regular*.

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For M being the adjacency matrix, this happens when Λ is *net-regular*.

For M being the Laplacian matrix, this happens when Λ is *negatively-regular*

Net-regular Signed Graphs

Definition

Given any signed graph Γ*, the* signed degree *of a vertex* v *in a signed graph is*

$$\mathrm{sdeg}(v) = d^{+}(v) - d^{-}(v).$$

where $d^{+}(v)$ *(resp.* $d^{-}(v)$*) is the number of incident positive (resp. negative) edges.*

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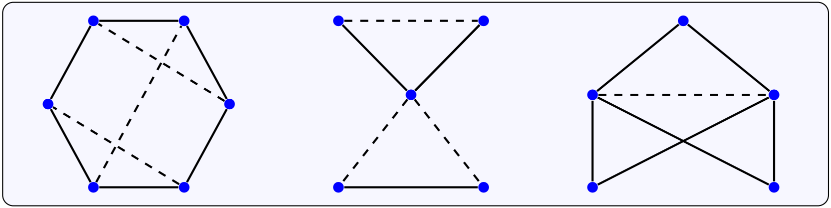
Definition

A signed graph is said to be net-regular if the vertices all have the same signed-degree.

Examples of Net-regular Signed Graphs



Λ Λ' Λ''

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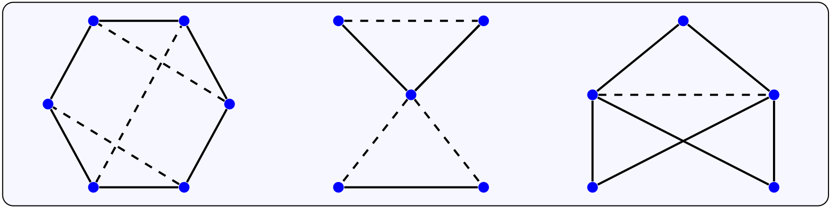

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$\mathrm{sdeg}(v) = 1$

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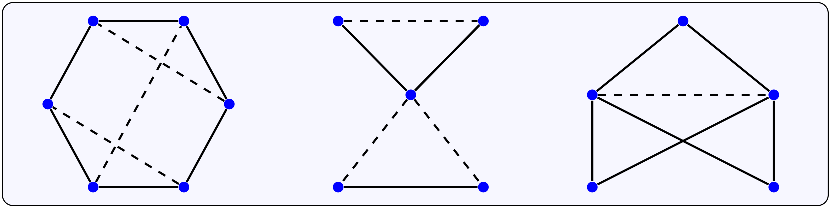

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$\mathrm{sdeg}(v) = 1$ $\mathrm{sdeg}(v) = 0$

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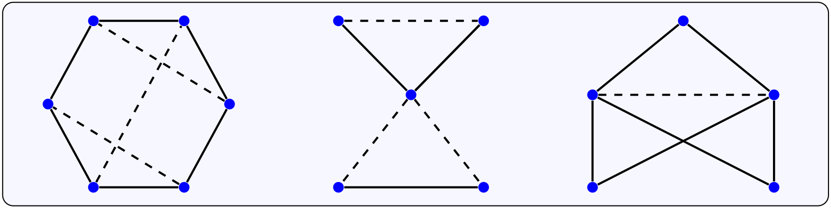

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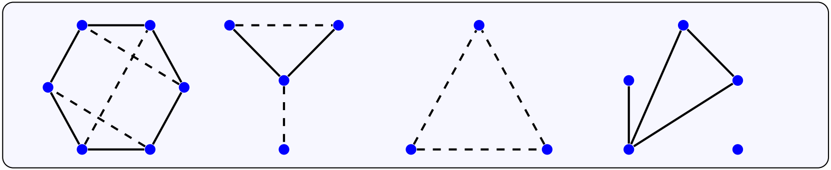
$d^-(v)$ is the same for all $v \in V(\Lambda)$

Examples of Negatively-regular Signed Graphs



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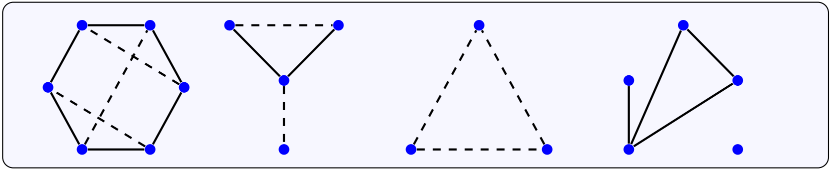

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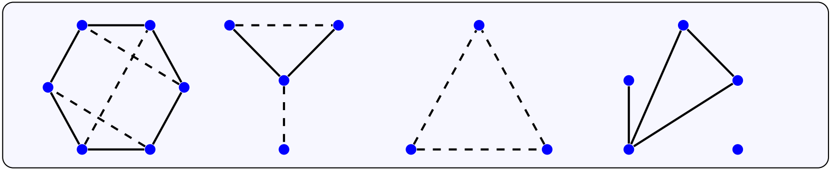

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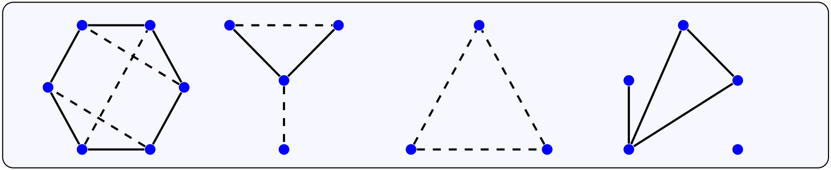

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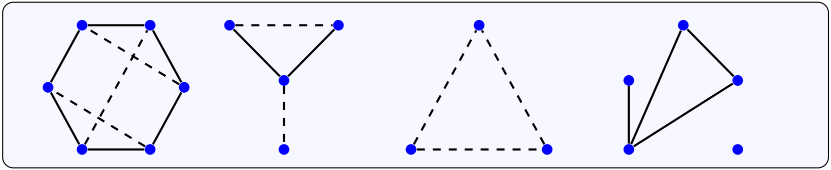

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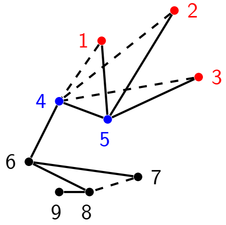
Restrictions on signed graph

The (c, s)-cluster on the signed graph Γ must be *homogeneous*



G

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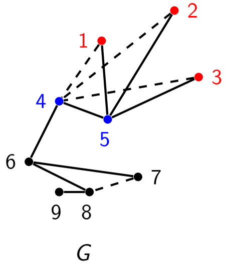
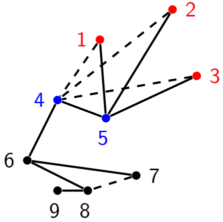


All edges connecting a fixed vertex of S to its neighbors in C are equally signed

Main result



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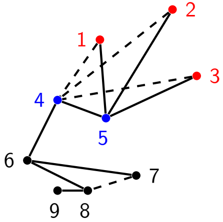

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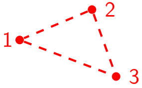

Λ

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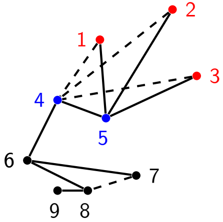
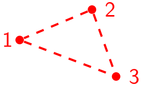
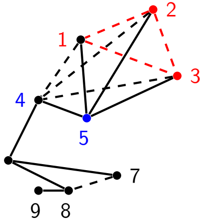


Γ



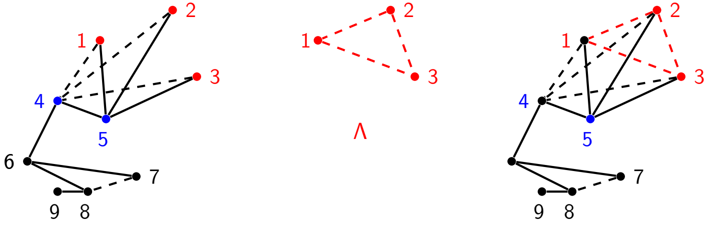
Λ



$\Gamma(\Lambda)$

$$M(\Lambda)\mathbf{1}_c = \mu_c(M(\Lambda))\mathbf{1}_c$$

Main result



Γ $\quad M(\Lambda)\mathbf{1}_c = \mu_c(M(\Lambda))\mathbf{1}_c$ $\quad \Gamma(\Lambda)$

$$\det(\lambda I - M(\Gamma(\Lambda))) = g_M^{\Lambda} \prod_{i=1}^{c-1} (\lambda - (s\delta + \mu_i(M(\Lambda)))).$$

where

$$g_M^{\Lambda} = p_{M(\Gamma)}(\lambda) - (\mu_c(M(\Lambda)) + s\delta) q_{M(\Gamma)}(\lambda).$$

Applications

Let $\mathcal{H}_A^{c,k}$ (resp. $\mathcal{H}_L^{c,k}$) be the set of all signed graphs of order c which are net-regular (resp. negatively regular) and

$$\mathrm{sdeg}(v) = k \qquad (\text{resp. } d^-(v) = k)$$

Theorem

Let Γ be a signed graph of order n having a homogeneous (c, s)-cluster (C, S).

Whatever Λ and Λ' we choose in $\mathcal{H}_A^{c,k}$ (resp. $\mathcal{H}_L^{c,k}$), the graphs $\Gamma(\Lambda)$ and $\Gamma(\Lambda')$ share a same set of $n - c + 1$ adjacency (resp. Laplacian) eigenvalues.

Graphs, groups, and more

Koper, May 28–June 1, 2018

Thank you!

(This is really THE END)

Applications

Theorem

Let Γ be a graph of order n with $h \geq 1$ pairwise disjoint homogeneous clusters $(C_1, S_1), \ldots, (C_h, S_h)$ such that

$$|C_i| = c_i \qquad \text{and} \qquad |S_1| = \cdots = |S_h| = s.$$

Then the multiplicity of the L-eigenvalue s satisfies

$$m_L(s) \geq \left(\sum_{j=1}^{h} c_j \right) - h.$$

Applications

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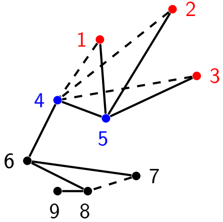
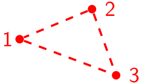
Proof: Use our theorem for Λ being the empty graph cK_1 ...

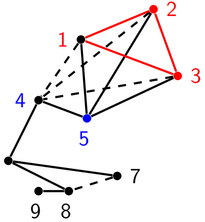
Example



Γ

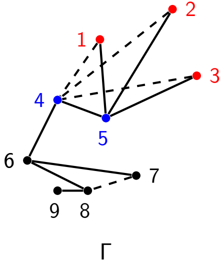


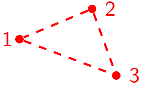
$\Lambda = C_3^-$

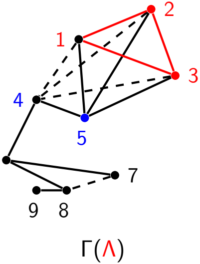


$\Gamma(\Lambda)$

Example

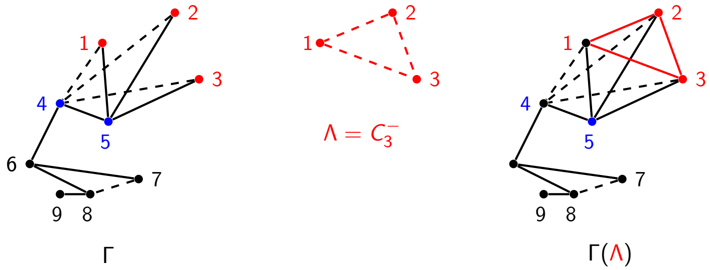






$$\mu_1(A(C_3^-)) = \mu_2(A(C_3^-)) = 1, \ \mu_3(A(C_3^-)) = -2$$

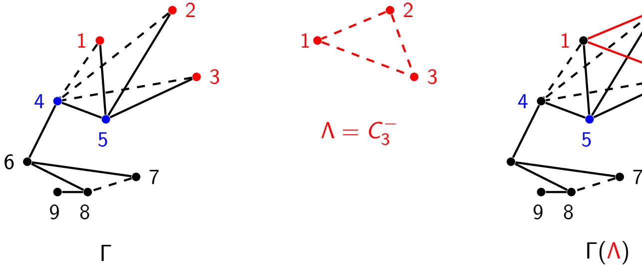
Example



$$\mu_1(A(C_3^-)) = \mu_2(A(C_3^-)) = 1,\ \mu_3(A(C_3^-)) = -2$$

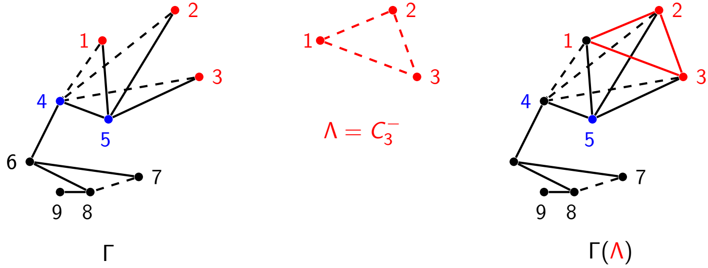
$$m_{A(\Gamma(C_3^-))}(1) \geq 2$$

Example



$$\mu_1(L(C_3^-)) = \mu_2(L(C_3^-)) = 1, \ \mu_3(L(C_3^-)) = 4$$

Example



$$\mu_1(L(C_3^-)) = \mu_2(L(C_3^-)) = 1, \; \mu_3(L(C_3^-)) = 4$$

$$m_{L(\Gamma(C_3^-))}(3) = m_{L(\Gamma(C_3^-))}(s + \mu_1(L(C_3^-))) \geq 2$$