Twofold triple systems that disprove Tutte's conjecture:

Bipartite non-Hamiltonian 2-block intersection graphs

Rosalind Cameron and David Pike

Memorial University of Newfoundland

Koper, 2018

A (v, k, λ)-BIBD consists of a v-set V of elements (called points) together with a collection $\mathcal{B}$ of k-subsets (called blocks) of V such that each pair of points from V occurs in exactly λ blocks.

Definitions

A (v, k, λ)-BIBD consists of a v-set V of elements (called points) together with a collection $\mathcal{B}$ of k-subsets (called blocks) of V such that each pair of points from V occurs in exactly λ blocks.

- A $(v, 3, 1)$-BIBD is a STS(v).

A (v, k, λ)-BIBD consists of a v-set V of elements (called points) together with a collection $\mathcal{B}$ of k-subsets (called blocks) of V such that each pair of points from V occurs in exactly λ blocks.

- A $(v, 3, 1)$-BIBD is a STS(v).
- A $(v, 3, 2)$-BIBD is a TTS(v).

Example: TTS(7) (i.e. $(7, 3, 2)$-BIBD)

Definitions

A (v, k, λ)-BIBD consists of a v-set V of elements (called points) together with a collection $\mathcal{B}$ of k-subsets (called blocks) of V such that each pair of points from V occurs in exactly λ blocks.

- A $(v, 3, 1)$-BIBD is a STS(v).
- A $(v, 3, 2)$-BIBD is a TTS(v).

Example: TTS(7) (i.e. $(7, 3, 2)$-BIBD)
$V = \{1, 2, \ldots, 7\}$

Definitions

A (v, k, λ)-BIBD consists of a v-set V of elements (called points) together with a collection $\mathcal{B}$ of k-subsets (called blocks) of V such that each pair of points from V occurs in exactly λ blocks.

- A $(v, 3, 1)$-BIBD is a STS(v).
- A $(v, 3, 2)$-BIBD is a TTS(v).

Example: TTS(7) (i.e. $(7, 3, 2)$-BIBD)
$V = \{1, 2, \ldots, 7\}$

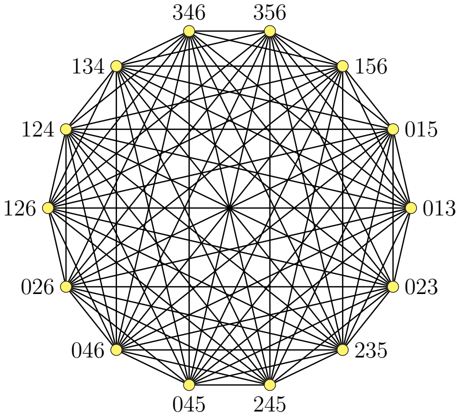
$$
\begin{array}{cc}
\{1,2,4\} & \{1,2,6\} \\
\{2,3,5\} & \{2,3,7\} \\
\{3,4,6\} & \{3,4,1\} \\
\{4,5,7\} & \{4,5,2\} \\
\{5,6,1\} & \{5,6,3\} \\
\{6,7,2\} & \{6,7,4\} \\
\{7,1,3\} & \{7,1,4\}
\end{array}
$$

Definitions

The block intersection graph (BIG) of a design $\mathcal{D}$ is the graph whose vertices are the blocks of $\mathcal{D}$ and two blocks B_1, B_2 in $\mathcal{D}$ are adjacent if $|B_1 \cap B_2| > 0$.

The block intersection graph (BIG) of a design $\mathcal{D}$ is the graph whose vertices are the blocks of $\mathcal{D}$ and two blocks B_1, B_2 in $\mathcal{D}$ are adjacent if $|B_1 \cap B_2| > 0$.

Example: BIG of TTS(7)

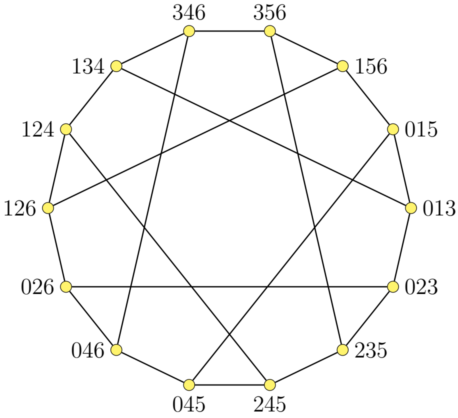


Definitions

The i-block intersection graph (i-BIG) of a design $\mathcal{D}$ is the graph whose vertices are the blocks of $\mathcal{D}$ and two blocks B_1, B_2 in $\mathcal{D}$ are adjacent if $|B_1 \cap B_2| = i$.

The i-block intersection graph (i-BIG) of a design $\mathcal{D}$ is the graph whose vertices are the blocks of $\mathcal{D}$ and two blocks B_1, B_2 in $\mathcal{D}$ are adjacent if $|B_1 \cap B_2| = i$.

Example: 2-BIG of TTS(7)



Background: BIGs and 1-BIGs

BIGs of (v, k, λ)-BIBDs:

Background: BIGs and 1-BIGs

BIGs of (v, k, λ)-BIBDs:

- Hamiltonian (Horák and Rosa, 1988)

Background: BIGs and 1-BIGs

BIGs of (v, k, λ)-BIBDs:

- Hamiltonian (Horák and Rosa, 1988)
- For $\lambda = 1$, $k \geq 3$: edge pancyclic (Alspach and Hare, 1991)
- Pancyclic (Mamut, Pike and Raines, 2004)
- Cycle extendable (Abueida and Pike, 2013)

Background: BIGs and 1-BIGs

BIGs of (v, k, λ)-BIBDs:

- Hamiltonian (Horák and Rosa, 1988)
- For $\lambda = 1$, $k \geq 3$: edge pancyclic (Alspach and Hare, 1991)
- Pancyclic (Mamut, Pike and Raines, 2004)
- Cycle extendable (Abueida and Pike, 2013)

Hamiltonian 1-BIGs

- $(v, k, 1)$-BIBD (Horák and Rosa, 1988)

Background: BIGs and 1-BIGs

BIGs of (v, k, λ)-BIBDs:

- Hamiltonian (Horák and Rosa, 1988)
- For $\lambda = 1$, $k \geq 3$: edge pancyclic (Alspach and Hare, 1991)
- Pancyclic (Mamut, Pike and Raines, 2004)
- Cycle extendable (Abueida and Pike, 2013)

Hamiltonian 1-BIGs

- $(v, k, 1)$-BIBD (Horák and Rosa, 1988)
- $(v, 3, \lambda)$-BIBD with $v \geq 12$ (Horák, Pike and Raines, 1999)
- $(v, 4, \lambda)$-BIBD with $v \geq 136$ (Jesso, Pike and Shalaby, 2011)
- $(v, 5, \lambda)$-BIBD with $v \geq 305$ (Jesso, 2011)

Background

Hamilton cycle in 1-BIG of a STS is a minimal change ordering of the blocks.

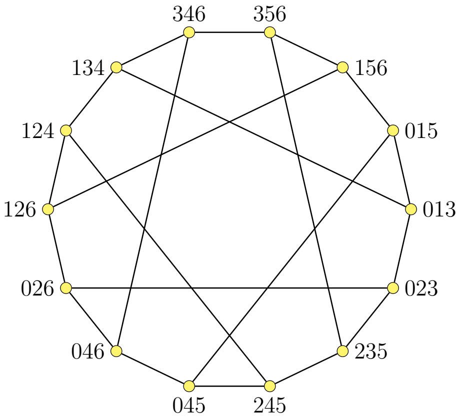
Background

Hamilton cycle in 2-BIG of a TTS is a minimal change ordering of the blocks.

Background

Hamilton cycle in 2-BIG of a TTS is a minimal change ordering of the blocks.

Example: 2-BIG of TTS(7)



Background: 2-BIGs

Some results for 2-BIGs of TTS(v).

- $v \geq 4$ such that $v \equiv 0, 1 \mod 3$ and $v \neq 6$, there exists a TTS(v) whose 2-BIG is Hamiltonian. (Dewar and Stevens; Erzurumluoğlu and Pike)

Background: 2-BIGs

Some results for 2-BIGs of TTS(v).

- $v \geq 4$ such that $v \equiv 0, 1 \mod 3$ and $v \neq 6$, there exists a TTS(v) whose 2-BIG is Hamiltonian. (Dewar and Stevens; Erzurumluoğlu and Pike)
- $v = 6$ or $v > 12$ and $v \equiv 0, 1 \mod 3$, there exists a TTS(v) whose 2-BIG is non-Hamiltonian. (Erzurumluoğlu and Pike)

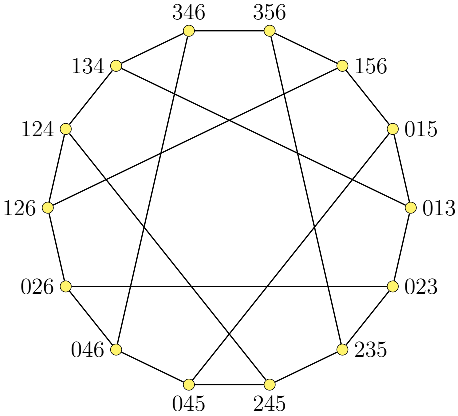
Background: 2-BIGs

Some results for 2-BIGs of TTS(v).

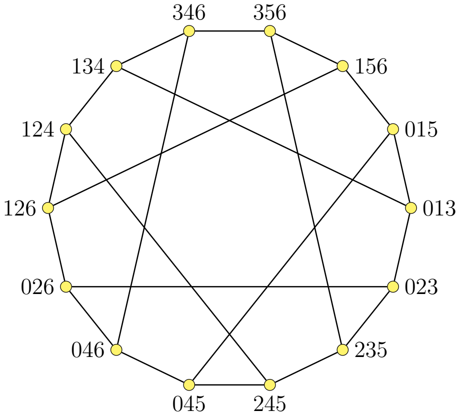
- $v \geq 4$ such that $v \equiv 0, 1 \mod 3$ and $v \neq 6$, there exists a TTS(v) whose 2-BIG is Hamiltonian. (Dewar and Stevens; Erzurumluoğlu and Pike)
- $v = 6$ or $v > 12$ and $v \equiv 0, 1 \mod 3$, there exists a TTS(v) whose 2-BIG is non-Hamiltonian. (Erzurumluoğlu and Pike)

Can we find sufficient conditions for Hamiltonian 2-BIG of TTS?

Background: 2-BIGs of TTS

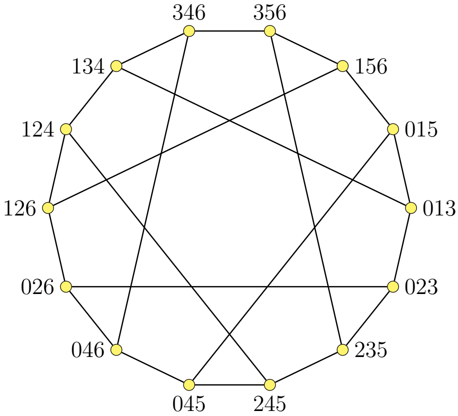


Background: 2-BIGs of TTS



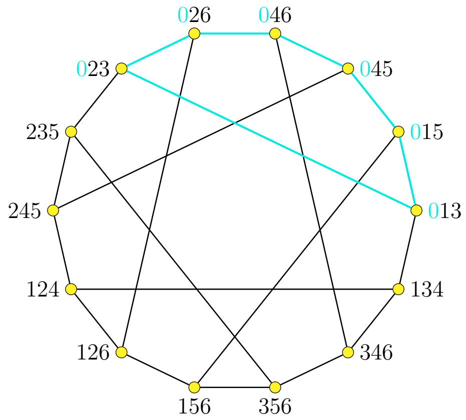
- cubic

Background: 2-BIGs of TTS



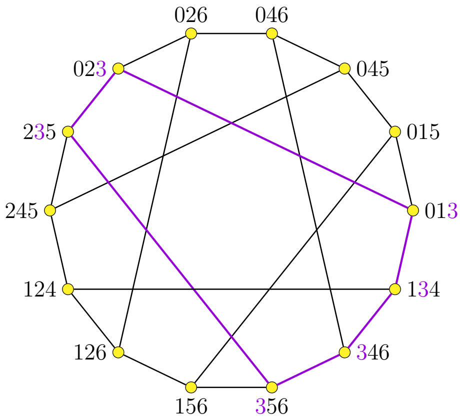
- cubic
- 3-connected (M. Colbourn and Johnstone, 1984)

Background: 2-BIGs of TTS



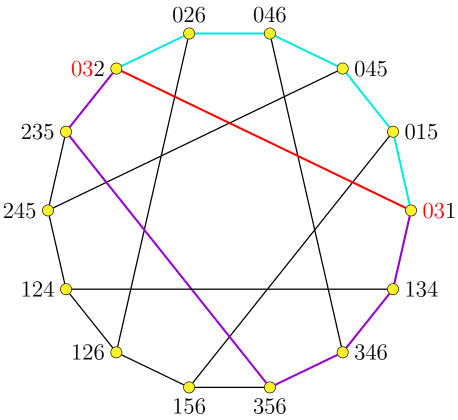
- cubic
- 3-connected (M. Colbourn and Johnstone, 1984)
- cycle double cover (2-BIG labelling)

Background: 2-BIGs of TTS



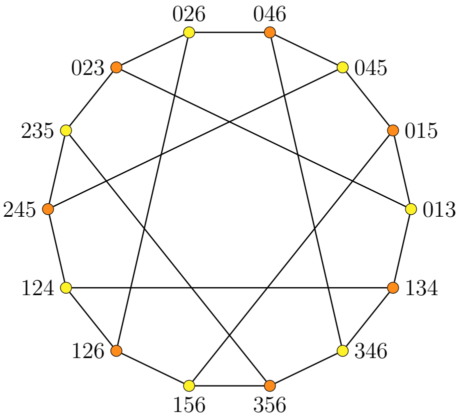
- cubic
- 3-connected (M. Colbourn and Johnstone, 1984)
- cycle double cover (2-BIG labelling)

Background: 2-BIGs of TTS



- cubic
- 3-connected (M. Colbourn and Johnstone, 1984)
- cycle double cover (2-BIG labelling)

Background: 2-BIGs of TTS(7)



- cubic
- 3-connected (M. Colbourn and Johnstone, 1984)
- cycle double cover (2-BIG labelling)
- bipartite

Background: 2-BIGs

Lemma

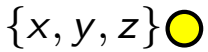
The 2-BIG of (partial) TTS is bipartite if and only if it can be partitioned into two (partial) STS.

Background: 2-BIGs

Lemma

The 2-BIG of (partial) TTS is bipartite if and only if it can be partitioned into two (partial) STS.

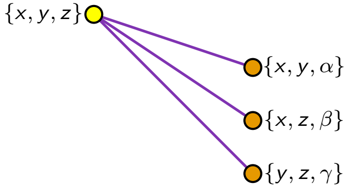
Proof.



LEMMA

The 2-BIG of (partial) TTS is bipartite if and only if it can be partitioned into two (partial) STS.

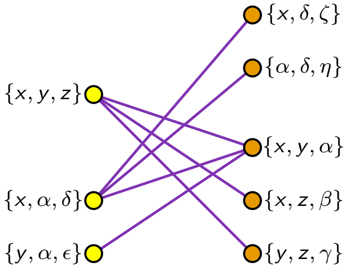
Proof.



LEMMA

The 2-BIG of (partial) TTS is bipartite if and only if it can be partitioned into two (partial) STS.

Proof.



Background: 2-BIGs

2-BIG of TTS: 3-connected cubic graph.

Background: 2-BIGs

2-BIG of TTS: 3-connected cubic graph.

Conjectures:

Background: 2-BIGs

2-BIG of TTS: 3-connected cubic graph.

Conjectures:

- Tait (1884): every planar 3-connected cubic graph is Hamiltonian.

Background: 2-BIGs

2-BIG of TTS: 3-connected cubic graph.

Conjectures:

- Tait (1884): every planar 3-connected cubic graph is Hamiltonian.
    - disproved by Tutte (1946).

Background: 2-BIGs

2-BIG of TTS: 3-connected cubic graph.

Conjectures:

- Tait (1884): every planar 3-connected cubic graph is Hamiltonian.
  - disproved by Tutte (1946).
- Tutte (1971): every bipartite 3-connected cubic graph is Hamiltonian.

Background: 2-BIGs

2-BIG of TTS: 3-connected cubic graph.

Conjectures:

- Tait (1884): every planar 3-connected cubic graph is Hamiltonian.
  - disproved by Tutte (1946).
- Tutte (1971): every bipartite 3-connected cubic graph is Hamiltonian.
  - Disproved by Horton (1970s).

Background: 2-BIGs

2-BIG of TTS: 3-connected cubic graph.

Conjectures:

- Tait (1884): every planar 3-connected cubic graph is Hamiltonian.
  - disproved by Tutte (1946).
- Tutte (1971): every bipartite 3-connected cubic graph is Hamiltonian.
  - Disproved by Horton (1970s).
- Barnette (1969): every bipartite planar 3-connected cubic graph is Hamiltonian.

Background: 2-BIGs

2-BIG of TTS: 3-connected cubic graph.

Conjectures:

- Tait (1884): every planar 3-connected cubic graph is Hamiltonian.
  - disproved by Tutte (1946).
- Tutte (1971): every bipartite 3-connected cubic graph is Hamiltonian.
  - Disproved by Horton (1970s).
- Barnette (1969): every bipartite planar 3-connected cubic graph is Hamiltonian.
  - Still open.

Preliminary results

Some observations:

- Counter-examples to Tutte's conjecture are not 2-BIGs of TTS.

Some observations:

- Counter-examples to Tutte's conjecture are not 2-BIGs of TTS.
- For $v \leq 13$: bipartite and connected $\implies$ Hamiltonian 2-BIG.

Preliminary results

Some observations:

- Counter-examples to Tutte's conjecture are not 2-BIGs of TTS.
- For $v \leq 13$: bipartite and connected $\implies$ Hamiltonian 2-BIG.
- Constructions for non-Hamiltonian 2-BIG $\implies$ not bipartite.

Preliminary results

Some observations:

- Counter-examples to Tutte's conjecture are not 2-BIGs of TTS.
- For $v \leq 13$: bipartite and connected $\implies$ Hamiltonian 2-BIG.
- Constructions for non-Hamiltonian 2-BIG $\implies$ not bipartite.

But...

Main result

Theorem (RC, Pike (2018+))

There exists an integer N such that for all admissible $v \geqslant N$, there is a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian. Furthermore, $13 < N \leqslant 663$.

Main result

Theorem (RC, Pike (2018+))

There exists an integer N such that for all admissible $v \geqslant N$, there is a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian. Furthermore, $13 < N \leqslant 663$.

Proof.

- Construct a TTS(331).

Main result

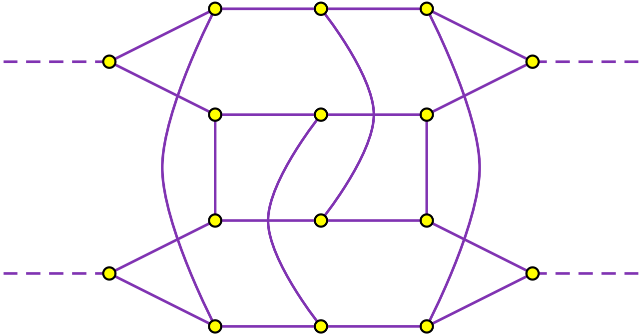
Theorem (RC, Pike (2018+))

There exists an integer N such that for all admissible $v \geqslant N$, there is a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian. Furthermore, $13 < N \leqslant 663$.

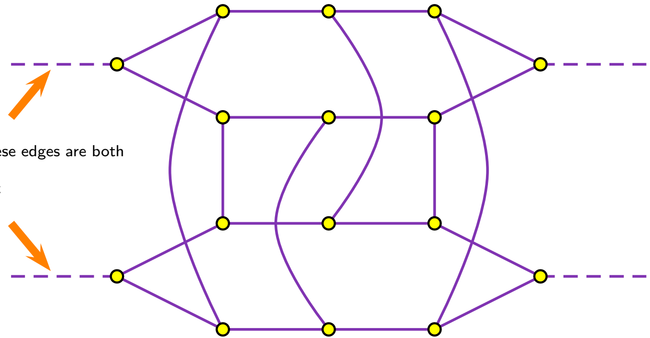
Proof.

- Construct a TTS(331).
- Embed TTS(u) in TTS(v) where $v > 2u$.

Main result: tts(331)

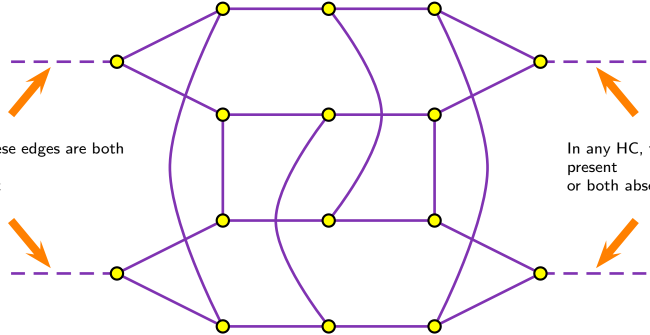


Main result: tts(331)



In any HC, these edges are both present
or both absent

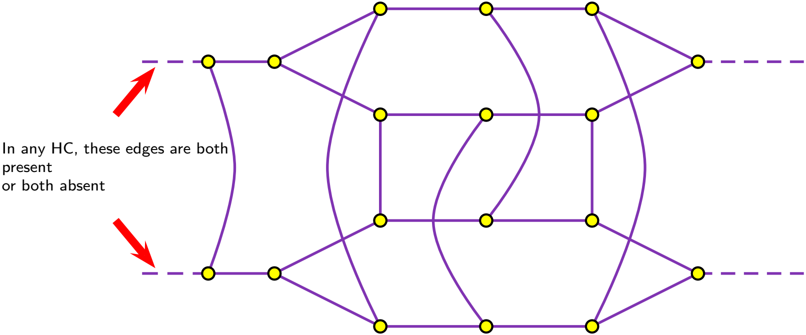
Main result: tts(331)



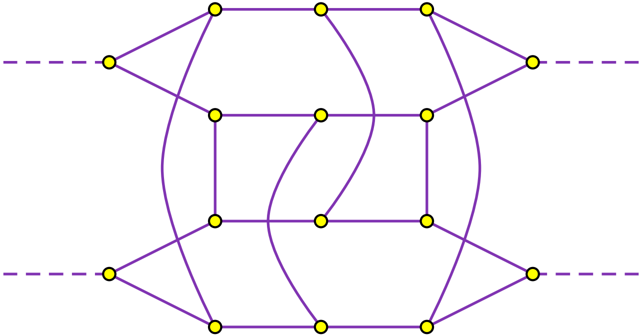
In any HC, these edges are both present
or both absent

In any HC, these edges are both present
or both absent

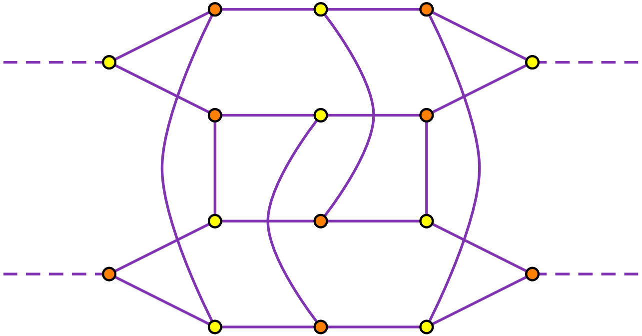
Main result: tts(331)



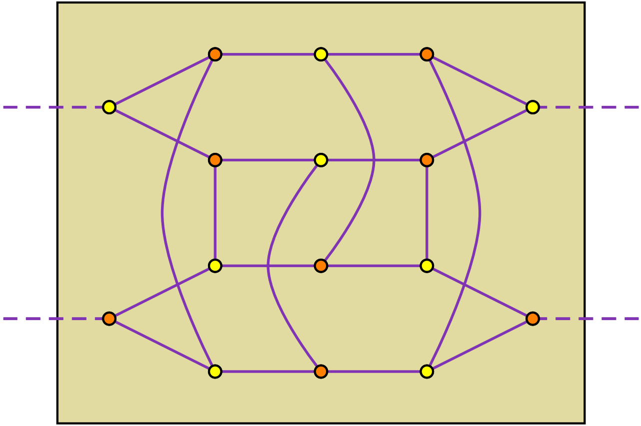
Main result: tts(331)



Main result: tts(331)

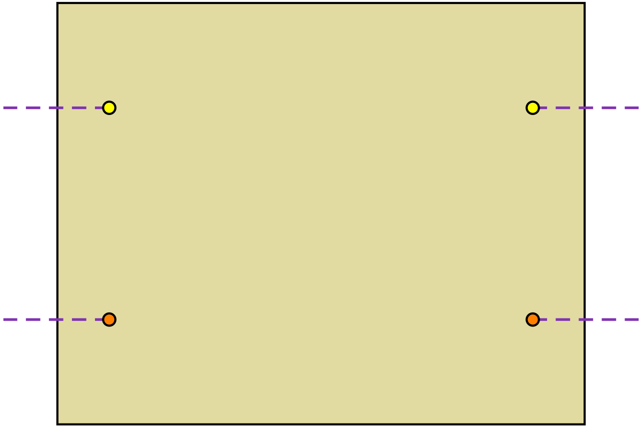


Configuration $\mathbb{T}$

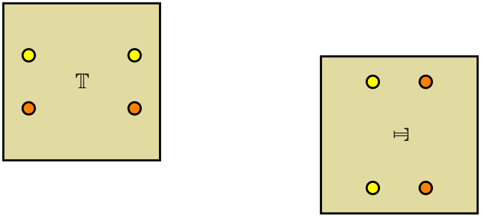


Main result: tts(331)

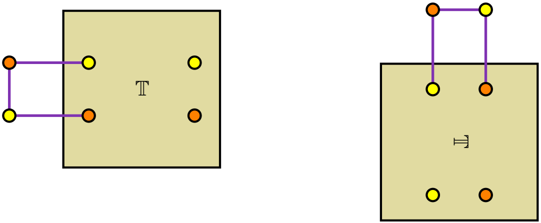
Configuration $\mathbb{T}$



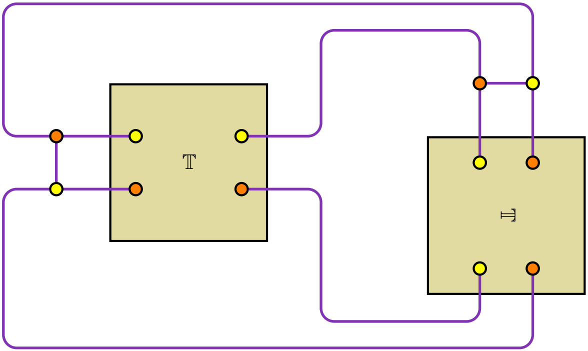
Main result: tts(331)



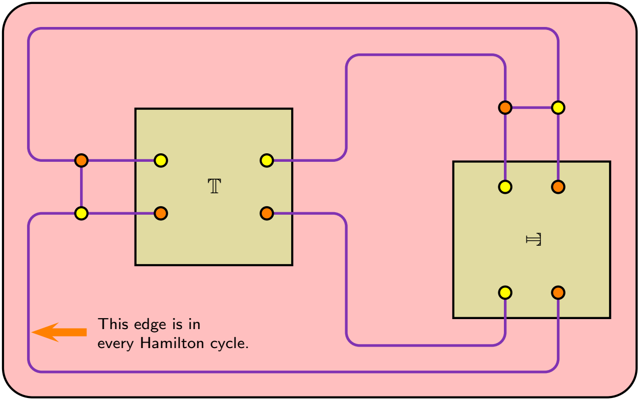
Main result: tts(331)



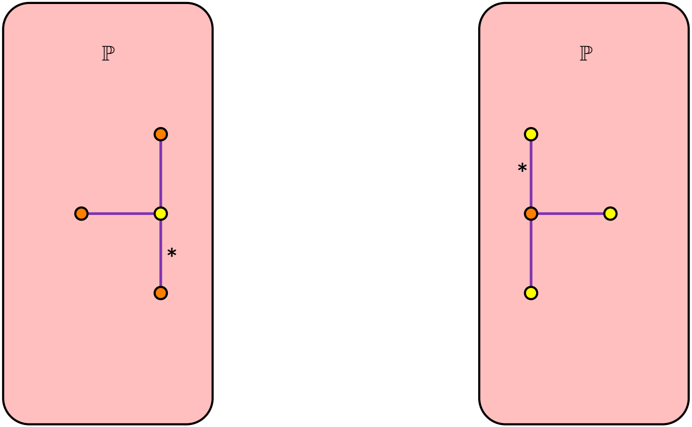
Main result: tts(331)



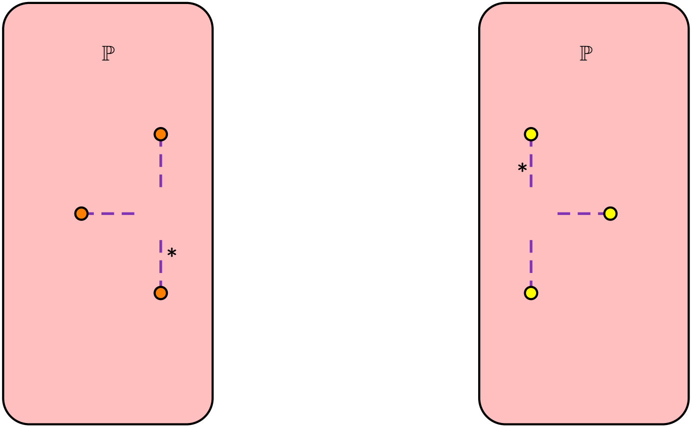
Configuration $\mathbb{P}$



Configuration $\mathbb{P} \asymp \mathbb{P}$

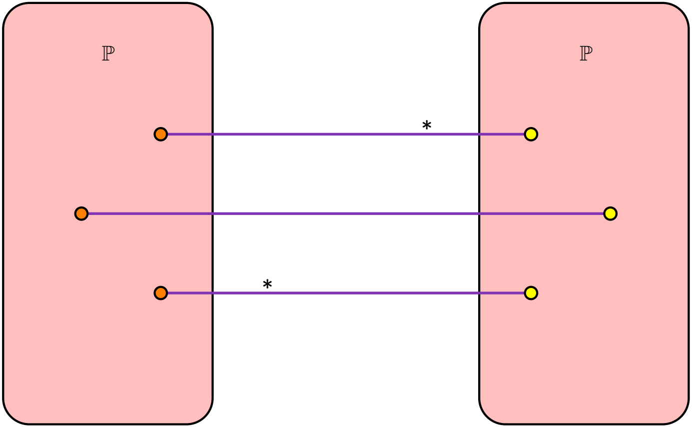


Configuration $\mathbb{P} \asymp \mathbb{P}$

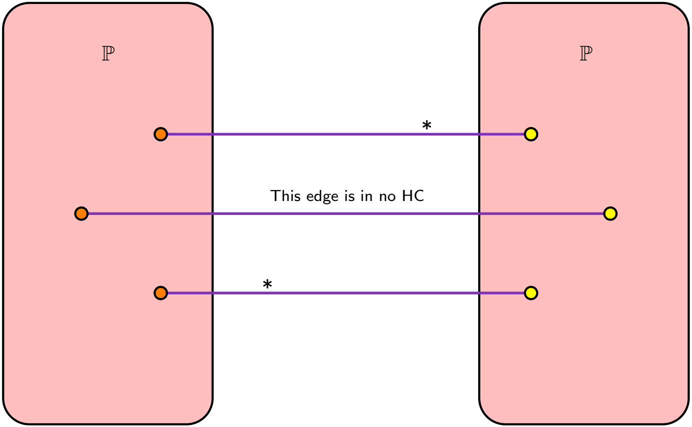


MAIN RESULT: TTS(331)

Configuration $\mathbb{P} \asymp \mathbb{P}$

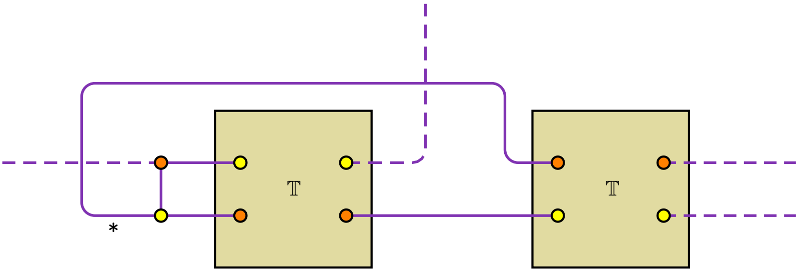


Configuration $\mathbb{P} \asymp \mathbb{P}$

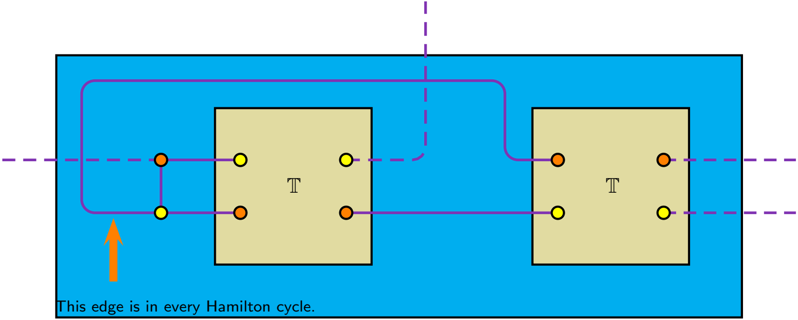


Main result: tts(331)

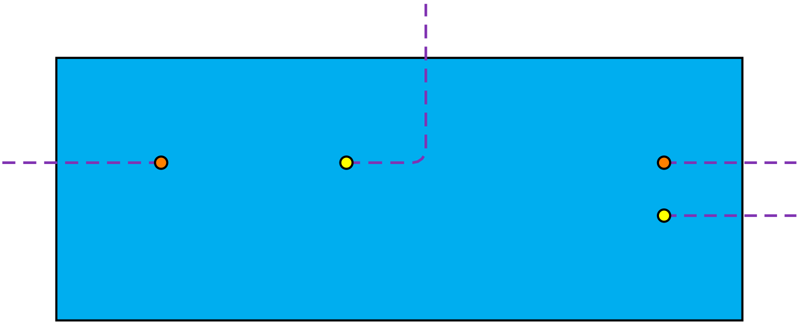
Configuration $\mathbb{X}$



Configuration $\mathbb{X}$

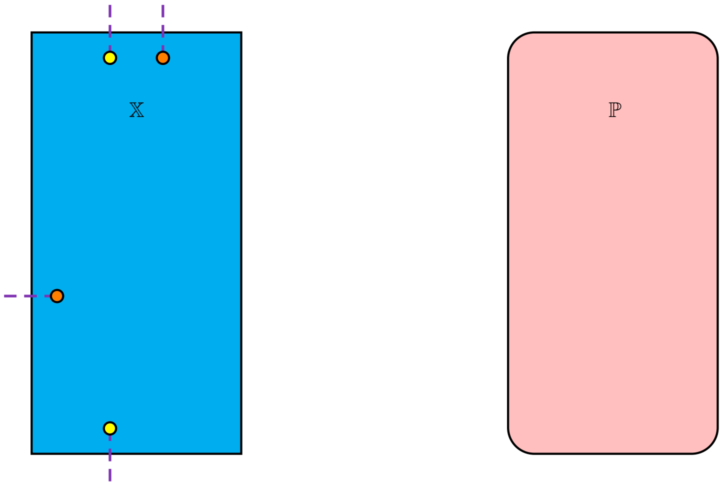


Configuration $\mathbb{X}$

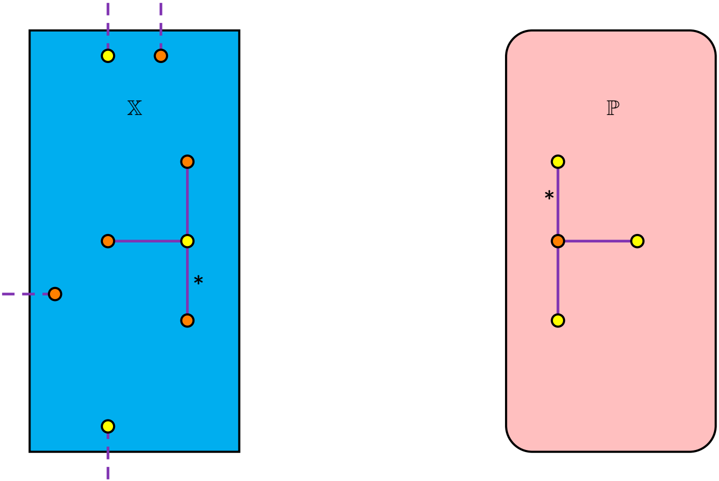


Main result: TTS(331)

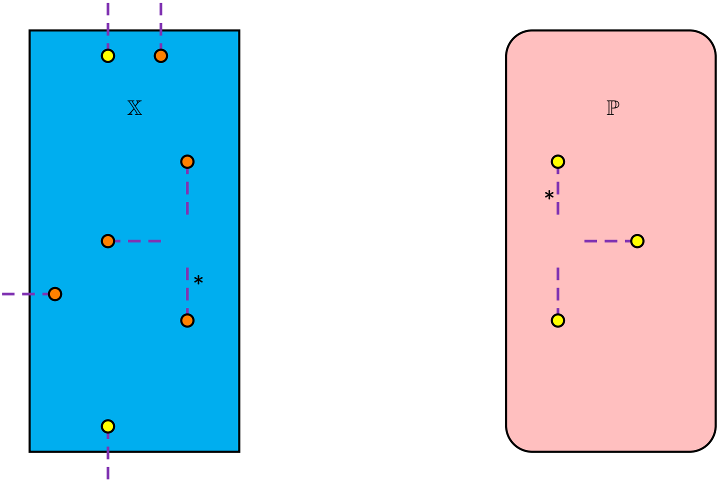
Configuration $\mathbb{X} \asymp \mathbb{P}$



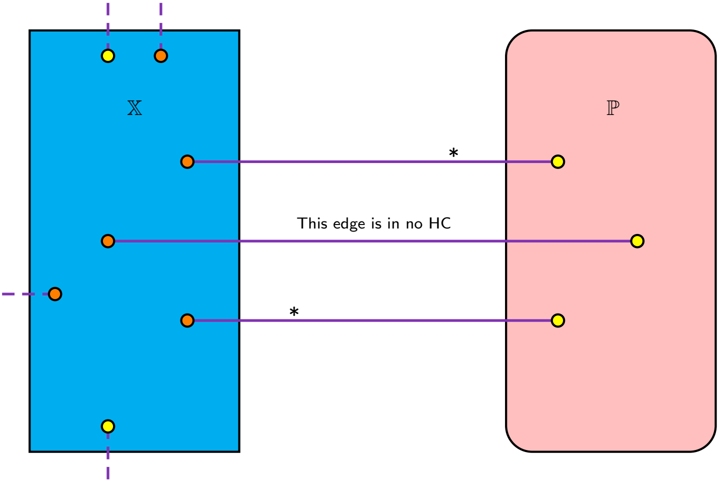
Configuration $\mathbb{X} \asymp \mathbb{P}$



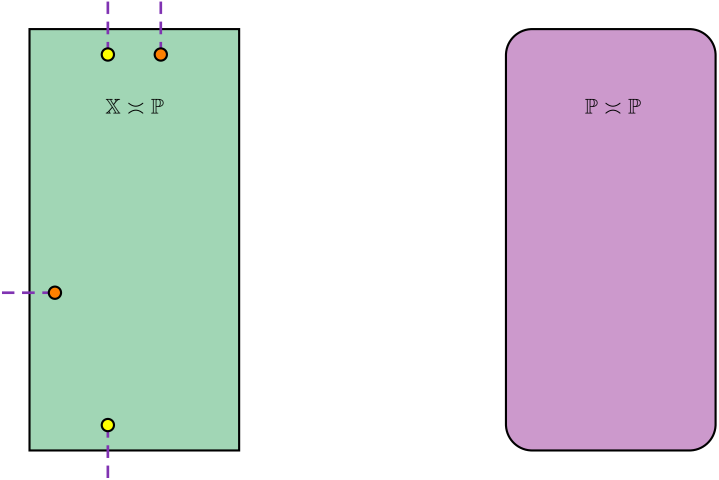
Configuration $\mathbb{X} \asymp \mathbb{P}$



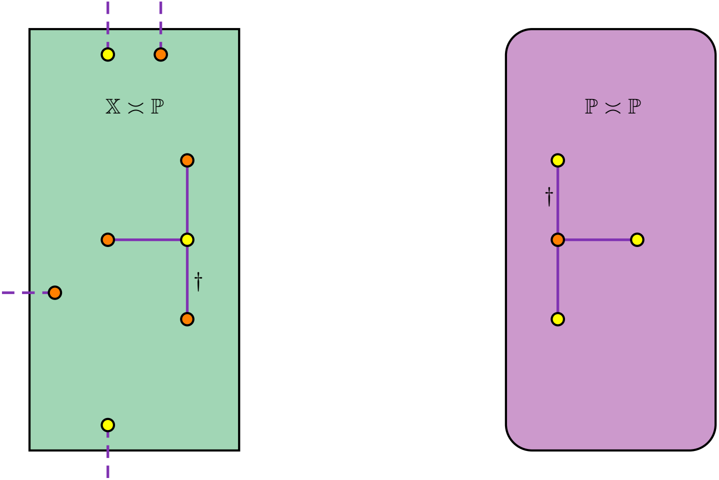
Configuration $\mathbb{X} \asymp \mathbb{P}$



Configuration $\mathbb{F}$: $(\mathbb{X} \asymp \mathbb{P}) \asymp (\mathbb{P} \asymp \mathbb{P})$

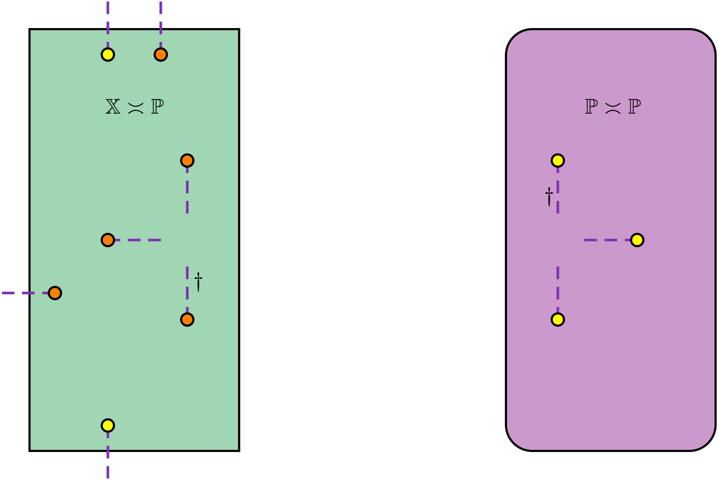


Configuration $\mathbb{F}$: $(\mathbb{X} \asymp \mathbb{P}) \asymp (\mathbb{P} \asymp \mathbb{P})$



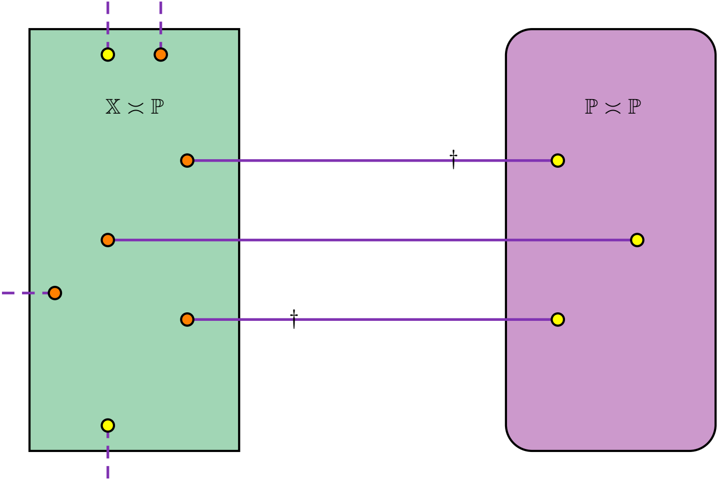
Main result: TTS(331)

Configuration $\mathbb{F}$: $(\mathbb{X} \asymp \mathbb{P}) \asymp (\mathbb{P} \asymp \mathbb{P})$



MAIN RESULT: TTS(331)

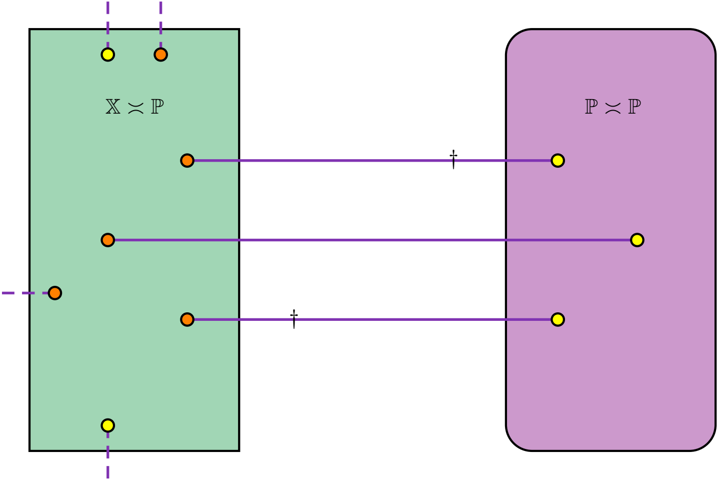
Configuration $\mathbb{F}$: $(\mathbb{X} \asymp \mathbb{P}) \asymp (\mathbb{P} \asymp \mathbb{P})$



Configuration $\mathbb{F}$ forbids Hamilton cycles

Main result: TTS(331)

Configuration $\mathbb{F}$: $(\mathbb{X} \asymp \mathbb{P}) \asymp (\mathbb{P} \asymp \mathbb{P})$



Configuration $\mathbb{F}$ forbids Hamilton cycles
Labelled by partial TTS(55)

Main result: tts(331)

Theorem (Lindner (1980))

Let (U, P_1) and (U, P_2) be partial STS(u). Then for every admissible $v \geq 6u + 1$, there exists a pair of STS(v) (V, S_1) and (V, S_2) such that (U, P_1) is embedded in (V, S_1), (U, P_2) is embedded in (V, S_2) and $P_1 \cap P_2 = S_1 \cap S_2$.

Main result: TTS(331)

Theorem (Lindner (1980))

Let (U, P_1) and (U, P_2) be partial STS(u). Then for every admissible $v \geq 6u + 1$, there exists a pair of STS(v) (V, S_1) and (V, S_2) such that (U, P_1) is embedded in (V, S_1), (U, P_2) is embedded in (V, S_2) and $P_1 \cap P_2 = S_1 \cap S_2$.

Configuration $\mathbb{F}$: partial TTS(55).

Main result: tts(331)

Theorem (Lindner (1980))

Let (U, P_1) and (U, P_2) be partial STS(u). Then for every admissible $v \geq 6u + 1$, there exists a pair of STS(v) (V, S_1) and (V, S_2) such that (U, P_1) is embedded in (V, S_1), (U, P_2) is embedded in (V, S_2) and $P_1 \cap P_2 = S_1 \cap S_2$.

Configuration $\mathbb{F}$: partial TTS(55).

Embed $\mathbb{F}$ in TTS(331).

THEOREM (RC, PIKE (2018+))

Suppose u and v are admissible integers such that $v > 2u$. If there exists a TTS(u) whose 2-BIG is bipartite connected and non-Hamiltonian, then there exists a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian.

Main result: embedding TTS

Theorem (RC, Pike (2018+))

Suppose u and v are admissible integers such that $v > 2u$. If there exists a TTS(u) whose 2-BIG is bipartite connected and non-Hamiltonian, then there exists a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian.

- partial STS(v) from difference triples
- 1-factorisations of circulant graphs
- Stern and Lenz (1980)

Open problems

Theorem (RC, Pike (2018+))

There exists an integer N such that for all admissible $v \geqslant N$, there is a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian. Furthermore, $13 < N \leqslant 663$.

Open problems

Theorem (RC, Pike (2018+))

There exists an integer N such that for all admissible $v \geqslant N$, there is a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian. Furthermore, $13 < N \leqslant 663$.

- What is the smallest admissible v such that there exists a TTS(v) with a connected bipartite non-Hamiltonian 2-BIG?

Open problems

Theorem (RC, Pike (2018+))

There exists an integer N such that for all admissible $v \geqslant N$, there is a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian. Furthermore, $13 < N \leqslant 663$.

- What is the smallest admissible v such that there exists a TTS(v) with a connected bipartite non-Hamiltonian 2-BIG?
  - $13 < v < 331$

Open problems

Theorem (RC, Pike (2018+))

There exists an integer N such that for all admissible $v \geqslant N$, there is a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian. Furthermore, $13 < N \leqslant 663$.

- What is the smallest admissible v such that there exists a TTS(v) with a connected bipartite non-Hamiltonian 2-BIG?
  - $13 < v < 331$
- What is the smallest integer N such that for all admissible $v > N$ there exists a TTS(v) with a connected bipartite non-Hamiltonian 2-BIG?

Open problems

Theorem (RC, Pike (2018+))

There exists an integer N such that for all admissible $v \geqslant N$, there is a TTS(v) whose 2-BIG is bipartite connected and non-Hamiltonian. Furthermore, $13 < N \leqslant 663$.

- What is the smallest admissible v such that there exists a TTS(v) with a connected bipartite non-Hamiltonian 2-BIG?
  - $13 < v < 331$
- What is the smallest integer N such that for all admissible $v > N$ there exists a TTS(v) with a connected bipartite non-Hamiltonian 2-BIG?
- For $v > 12$, find sufficient conditions for a TTS(v) to have a Hamiltonian 2-BIG.