Symmetry properties of generalized graph truncations

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Graphs, groups and more

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Joint work with Eduard Eiben and Robert Jajcay

A well-known example

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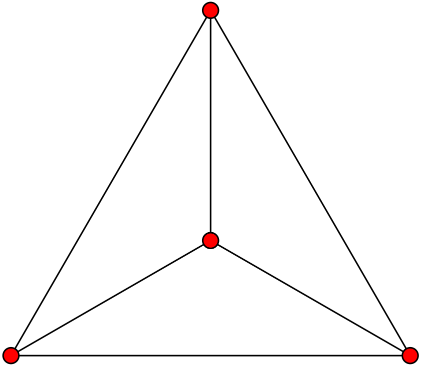

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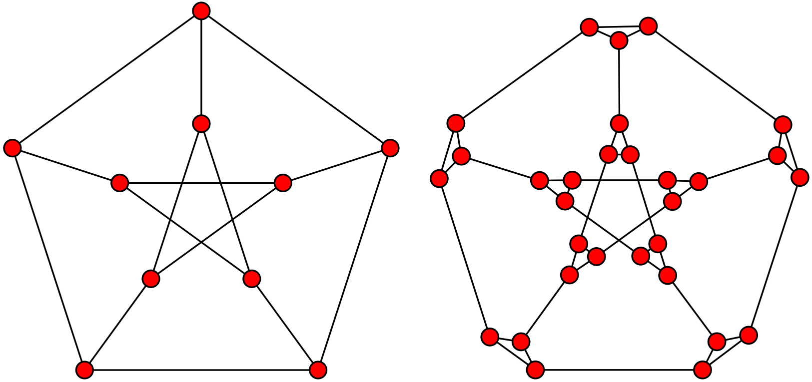

How to generalize the concept of truncations?

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- Very natural for maps (graphs embedded on a surface). Here vertices are replaced by cycles.
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- Sachs (1963): replace vertices by cycles.
- Exoo, Jajcay (2012): replace vertices by graphs of the correct order.
- One needs to prescribe (for each vertex) hot to do this.

The definition

- Γ a finite k-regular graph.
- Υ a graph of order k with $V(\Upsilon) = \{v_1, v_2, \ldots, v_k\}$.

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    - for each $u \in V(\Gamma)$ the restriction of ρ to $\{(u, w)\colon w \in \Gamma(u)\}$ is a bijection. ($D(\Gamma)$ is the set of darts (or arcs) of Γ.)

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- The *generalized graph truncation* $T(\Gamma, \rho; \Upsilon)$ has:
  - vertex-set $\{(u, v_i)\colon u \in V(\Gamma), 1 \leq i \leq k\}$;
  - edge-set is a union of two sets:

$\{(u, v_i)(u, v_j)\colon u \in V(\Gamma), v_i v_j \in E(\Upsilon)\}$ (red edges)
$\{(u, v_{\rho(u,w)})(w, v_{\rho(w,u)})\colon uw \in E(\Gamma)\}$ (blue edges).

Two examples

- In each of them $\Gamma = K_5$ with $V(\Gamma) = \{a, b, c, d, e\}$.
- In each of them $\Upsilon = C_4$ with $V(\Upsilon) = \{1, 2, 3, 4\}$ and $1 \sim 2, 4$.

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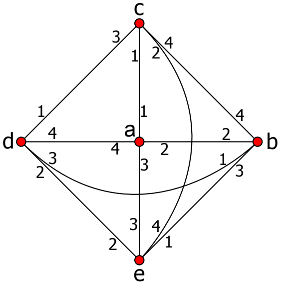
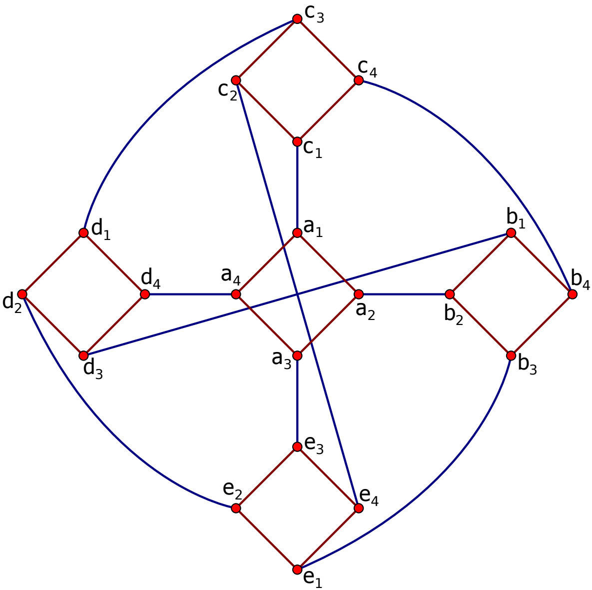
- We take two different vertex-neighborhood labellings.
- Simplify notation: for instance $(d, 3)$ is denoted by d_3.
- Are the two obtained graphs different?

The first example





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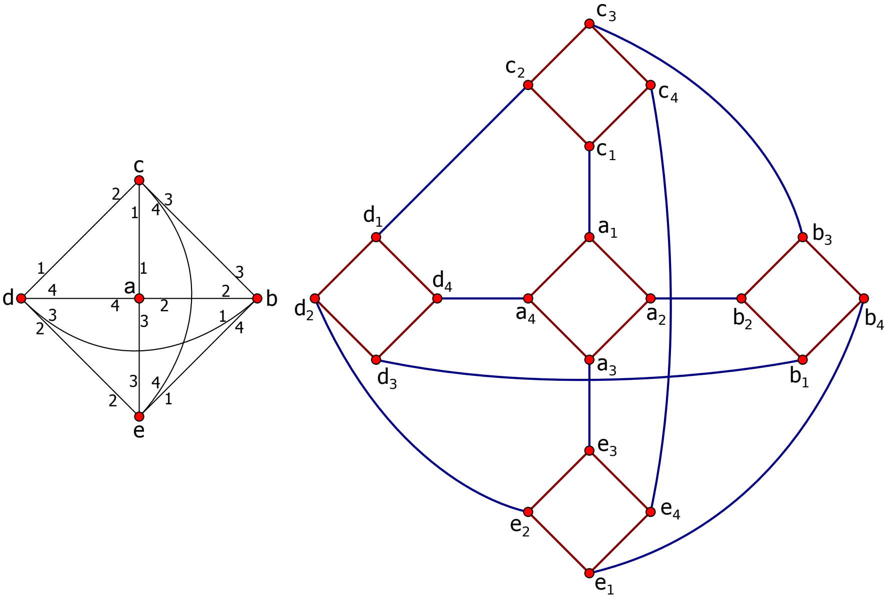

First observations

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Lemma (Exoo, Jajcay, 2012)

Let Γ be a k-regular graph and Υ a graph of order k and girth g. Then for any vertex-neighborhood labeling $\rho\colon D(\Gamma) \to \{1, 2, \ldots, k\}$ of Γ the shortest cycle of $T(\Gamma, \rho; \Upsilon)$ containing a blue edge is of length at least $2g$.

Symmetries of the truncation

- Let $\tilde{\Gamma} = T(\Gamma, \rho; \Upsilon)$.
- Let $\mathcal{P}_\Gamma = \{\{(u, v_i) \colon i \in \{1, 2, \ldots, k\}\} \colon u \in V(\Gamma)\}$ be the *natural partition* of $V(\tilde{\Gamma})$.

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Proposition (Eiben, Jajcay, Š)

Let $\tilde{\Gamma} = T(\Gamma, \rho; \Upsilon)$ be a generalized truncation and let $\tilde{G} \leq \mathrm{Aut}(\tilde{\Gamma})$ be any subgroup leaving $\mathcal{P}_\Gamma$ invariant. Then $\tilde{G}$ induces a natural faithful action on Γ and is thus isomorphic to a subgroup of $\mathrm{Aut}(\Gamma)$.

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- If $g \in \mathrm{Aut}(\Gamma)$ is a projection of some $\tilde{g} \in \mathrm{Aut}(\tilde{\Gamma})$, then $\tilde{g}$ is uniquely defined.
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  - $\tilde{g}$ is the *lift* of g.
- There can be *mixers* in $\mathrm{Aut}(\tilde{\Gamma})$.
- There can be elements of $\mathrm{Aut}(\Gamma)$ without lifts.

Symmetries that lift and symmetries that project

Corollary (Eiben, Jajcay, Š)

Let Γ be a k-regular graph of girth g, and $\tilde{\Gamma} = T(\Gamma, \rho; \Upsilon)$ be a generalized truncation with Υ connected and each of its edges lying on at least one cycle of length smaller than $2g$. Then the entire automorphism group $\mathrm{Aut}(\tilde{\Gamma})$ projects injectively onto a subgroup of $\mathrm{Aut}(\Gamma)$.

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Corollary (Eiben, Jajcay, Š)

Let Υ be a connected Cayley graph $\mathrm{Cay}(G; S)$ satisfying the property that S contains at least three elements out of which at least one belongs to the center $Z(G)$, and let $\tilde{\Gamma} = T(\Gamma, \rho; \Upsilon)$ be a generalized truncation. Then the entire automorphism group $\mathrm{Aut}(\tilde{\Gamma})$ projects injectively onto a subgroup of $\mathrm{Aut}(\Gamma)$.

Symmetries that lift and symmetries that project

Proposition (Eiben, Jajcay, Š)

Let $\tilde{\Gamma} = T(\Gamma, \rho; \Upsilon)$ *be a generalized truncation, and let* $g \in \mathrm{Aut}(\Gamma)$*. Then* g *lifts to* $\mathrm{Aut}(\tilde{\Gamma})$ *if and only if for every* $u \in V(\Gamma)$ *and each pair of its neighbors* w, x *we have*

$$v_{\rho(u,x)} \sim v_{\rho(u,w)} \iff v_{\rho(u^g,x^g)} \sim v_{\rho(u^g,w^g)} \text{ in } \Upsilon.$$

As a consequence, the set of all $g \in \mathrm{Aut}(\Gamma)$ *that lift to* $\mathrm{Aut}(\tilde{\Gamma})$ *is a subgroup of* $\mathrm{Aut}(\Gamma)$*.*

Construction from vertex-transitive graphs

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- Thus $(\Gamma(v), \mathcal{O}_v)$ is a graph with vertex set $\Gamma(v)$ and edge set $\mathcal{O}_v$.

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- Thus $(\Gamma(v), \mathcal{O}_v)$ is a graph with vertex set $\Gamma(v)$ and edge set $\mathcal{O}_v$.
- Define $T(\Gamma, G; \mathcal{O}_v)$ to be the graph with:
  - vertex set $\{(u, w) \colon u \in V(\Gamma), w \in \Gamma(u)\}$;
  - each (u, w) is adjacent to the vertex (w, u) and to all the vertices (u, w') for which there exists a $g \in G$ with the property $u^g = v$ and $\{w, w'\}^g \in \mathcal{O}_v$.

The first example revisited

- $\Gamma = K_5$ with $V(\Gamma) = \{a, b, c, d, e\}$.
- Take $G = \langle (a\,b\,c\,e\,d), (b\,c\,d\,e) \rangle$. Thus $G_a = \langle (b\,c\,d\,e) \rangle$.

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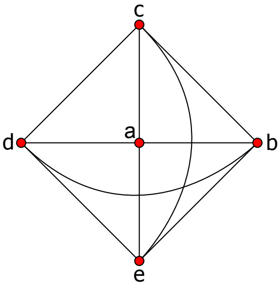
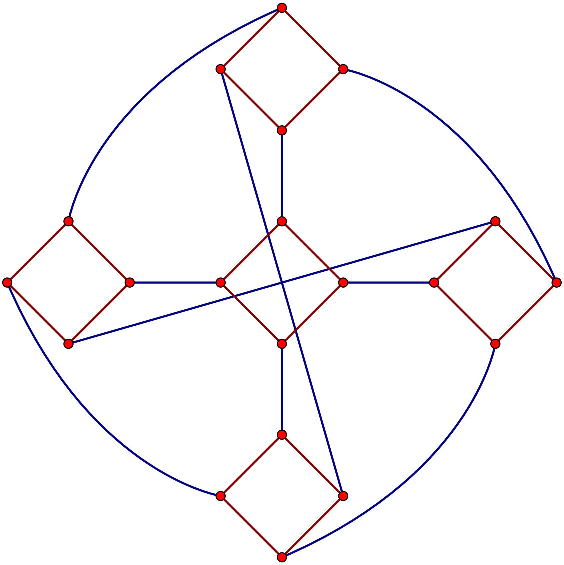
- $\Gamma = K_5$ with $V(\Gamma) = \{a, b, c, d, e\}$.
- Take $G = \langle (a\,b\,c\,e\,d), (b\,c\,d\,e) \rangle$. Thus $G_a = \langle (b\,c\,d\,e) \rangle$.
- G_a has two orbits on the 2-sets of $\Gamma(a) = \{b, c, d, e\}$, one of them being $\mathcal{O} = \{\{b, c\}, \{c, d\}, \{d, e\}, \{e, b\}\}$.
- It turns out that $T(K_5, G; \mathcal{O})$ is isomorphic to our first example.

The first example revisited





Construction from vertex-transitive graphs

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Proposition (Eiben, Jajcay, Š)

Let Γ, G, $\mathcal{O}_v$ and $\tilde{\Gamma} = T(\Gamma, G; \mathcal{O}_v)$ be as in the above construction. An automorphism h of Γ lifts to $\tilde{\Gamma}$ if and only if $\mathcal{O}_v$ is a union of orbits of the action of $\langle G, h \rangle_v$ on the 2-sets of elements from $\Gamma(v)$.

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Let Γ be an arc-transitive graph and let $G \leq \mathrm{Aut}(\Gamma)$ be arc-transitive. Let $v \in V(\Gamma)$, let $\mathcal{O}_v$ be a union of orbits of the action of G_v on the 2-sets of elements from $\Gamma(v)$, and let $\tilde{\Gamma} = T(\Gamma, G; \mathcal{O}_v)$. Then G lifts to $\tilde{G} \leq \mathrm{Aut}(\tilde{\Gamma})$ which acts vertex-transitively on $\tilde{\Gamma}$. Moreover, the natural partition $\mathcal{P}_\Gamma$ is an imprimitivity block system for the action of $\tilde{G}$.

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- For the previous example thus $\mathrm{Aut}(\tilde{\Gamma}) \cong \mathrm{AGL}_1(5)$ holds.

Vertex-transitive truncations

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Theorem (Eiben, Jajcay, Š)

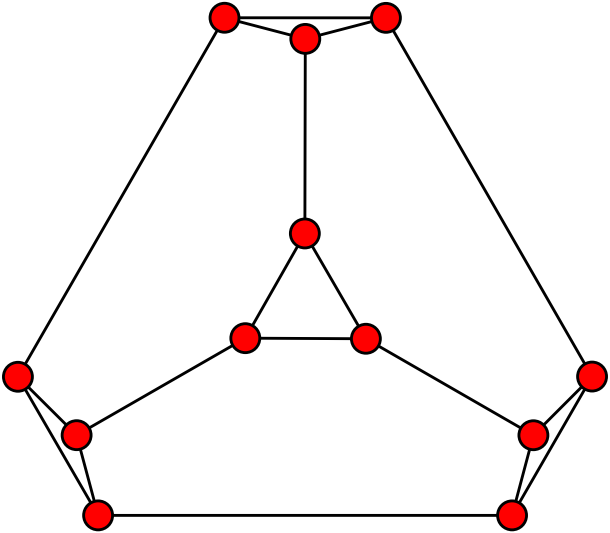
Let Γ be a vertex-transitive graph possessing a vertex-transitive group G of automorphisms admitting a nontrivial imprimitivity block system $\mathcal{B}$ on $V(\Gamma)$. If there exists a block $B \in \mathcal{B}$ with the property that each vertex of B has exactly one neighbor outside B and no two vertices of B have a neighbor in the same $B' \in \mathcal{B}$, $B' \neq B$, then Γ is a generalized truncation of an arc-transitive graph by a vertex-transitive graph in the sense of the Proposition.

Are all vertex-transitive truncations of this kind?

- Truncations of complete graphs by cycles (from AT graphs):

n	order	girth($\tilde{\Gamma}$)	$\|\mathrm{Aut}(\tilde{\Gamma})\|$	$\mathrm{Aut}(\tilde{\Gamma}) = \tilde{G}$
4	12	3	24	true
5	20	4	20	true
6	30	5	60	true
7	42	6	126	false
8	56	7	56	true
9	72	8	72	true
11	110	10	110	true
11	110	10	1320	false
13	156	9	156	true
13	156	9	156	true
17	272	11	272	true
17	272	11	272	true
17	272	12	272	true
17	272	13	272	true

The unique VT truncation of K_4 by C_3



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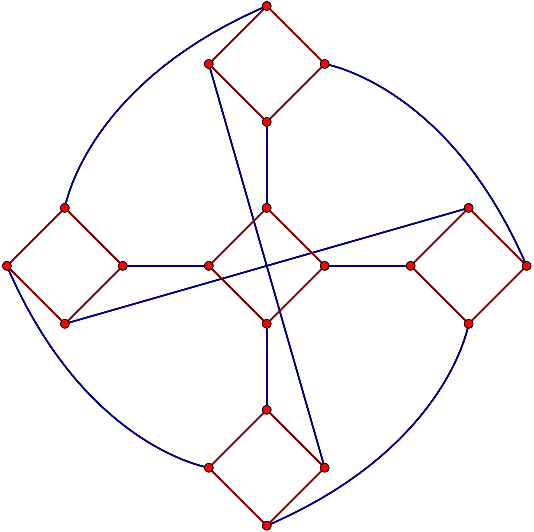

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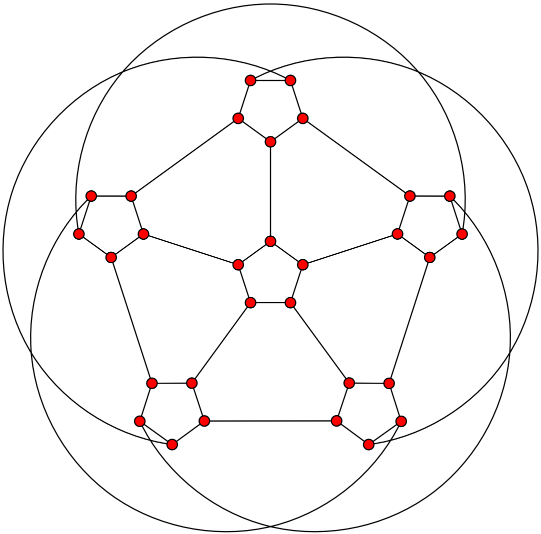
The unique VT truncation of K_6 by C_5



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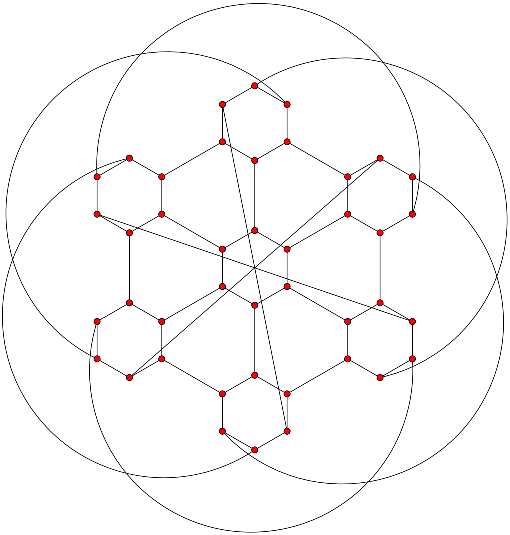
The unique VT truncation of K_7 by C_6



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- The other is the Cayley graph of the group

$$G = \langle\, a, b, c \mid a^2, b^2, c^2, acabcbcb, abcacbcacb, (ac)^6 \,\rangle$$

with respect to the connection set $S = \{a, b, c\}$.

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Problem

For each $n \geq 3$ classify all vertex-transitive generalized truncations of the complete graph K_n by the cycle C_{n-1}.

An application: cubic VT graphs of small girth

Theorem (Eiben, Jajcay, Š)

Let Γ *be a connected cubic graph of girth* 3*. Then* Γ *is vertex-transitive if and only if it is either the complete graph* K_4*, the prism* $\mathrm{Pr}(3)$*, or a (generalized) truncation of an arc-transitive cubic graph* Λ *by the* 3*-cycle* C_3*, in which case* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Lambda)$*.*

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Theorem (Eiben, Jajcay, Š)

Let Γ be a connected cubic graph of girth 4 and order $2n$. Then Γ is vertex-transitive if and only if it is isomorphic to the prism $\mathrm{Pr}(n)$ with $n \geq 4$, the Möbius ladder $\mathrm{Ml}(n)$ with $n \geq 3$, the generalized prism $\mathrm{GPr}(\frac{n}{2})$ (n even), or it is isomorphic to a generalized truncation of an arc-transitive tetravalent graph Λ by the 4-cycle C_4 in the sense of "the Theorem", in which case $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Lambda)$.

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Theorem (Eiben, Jajcay, Š)

Let Γ *be a connected cubic graph of girth* 5*. Then* Γ *is vertex-transitive if and only if it is either isomorphic to the Petersen graph or the Dodecahedron graph, or it is isomorphic to a generalized truncation of an arc-transitive* 5*-valent graph* Λ *by the* 5*-cycle* C_5 *in the sense of "the Theorem". In the latter case,* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Lambda)$*.*

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- Does not work this nicely for girths 6 and more.

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Theorem (Eiben, Jajcay, Š)

Let Γ be a vertex-transitive graph, let $G \leq \mathrm{Aut}(\Gamma)$ be vertex-transitive, and let $v \in V(\Gamma)$. Furthermore, let $v' \in V(\Gamma)$, let $x \in \mathrm{Aut}(\Gamma)$, and let $H = xGx^{-1}$. Then for any union $\mathcal{O}'$ of orbits of the action of the stabilizer $H_{v'}$ on the 2-sets of $\Gamma(v')$ there exists a union $\mathcal{O}$ of orbits of the action of the stabilizer G_v on the 2-sets of $\Gamma(v)$ such that $T(\Gamma, H; \mathcal{O}') \cong T(\Gamma, G; \mathcal{O})$. In particular, if $x \in (N_{\mathrm{Aut}(\Gamma)}(G))_v$ and $\mathcal{O}$ is a union of orbits of the action of G_v on the 2-sets of $\Gamma(v)$, then $T(\Gamma, G; \mathcal{O}) \cong T(\Gamma, G; \mathcal{O}^x)$.

Thank you!!!