# Some Variations on the Oberwolfach Theme

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University of Ottawa

*Graphs, Groups, and More:*
*Celebrating Brian Alspach's 80th and Dragan Marušič's 65th birthdays*
Koper, May 2018

# Outline

- Introduction to the Oberwolfach Problem
  - Terminology
  - The Oberwolfach Problem and the Spouse-Avoiding Variant
  - Current status of the Oberwolfach Problem
- Some variations on the Oberwolfach theme
  - The directed Oberwolfach Problem
  - The Spouse-Loving Variant
  - The Honeymoon Oberwolfach Problem

# Terminology: graph decompositions

- $\{H_1, H_2, \ldots, H_t\}$-decomposition of $G$,

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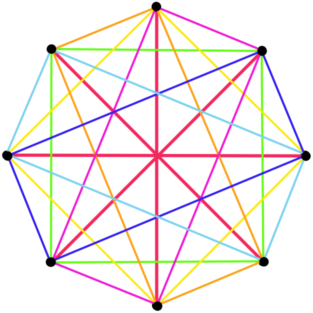
a partition of $E(G)$ into edge sets of its subgraphs $H_1, H_2, \ldots, H_t$



A decomposition of $K_8$ into six 4-cycles and a 1-factor

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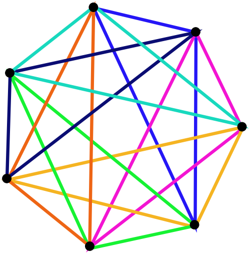

A $C_3$-decomposition of $K_7$

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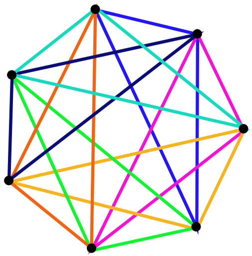

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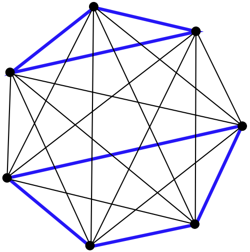

Figure: A $(C_3, C_4)$-factor in $K_7$.

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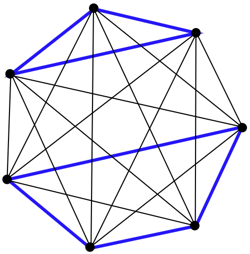

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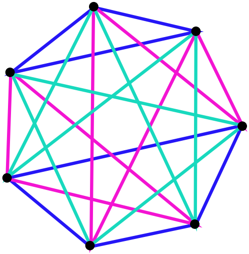

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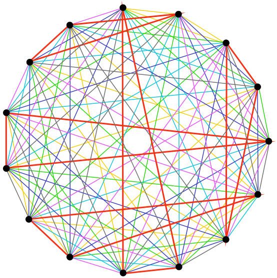

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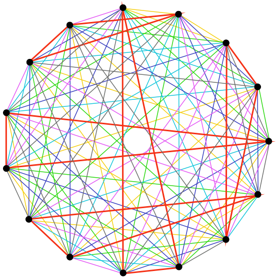

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# The Oberwolfach Problem

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  At a conference in Oberwolfach, $n = 2k + 1$ participants are to be seated at $t$ round tables for $k$ consecutive nights so that each participant sits next to each other participant **exactly once**. Can this be achieved with tables of sizes $m_1, m_2, \ldots, m_t$ assuming $m_1 + m_2 + \ldots + m_t = n$?

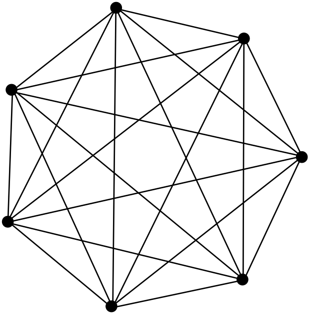

Figure: Oberwolfach Problem with $n = 7$, $m_1 = 3$, and $m_2 = 4$.

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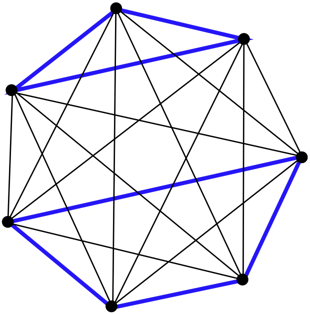

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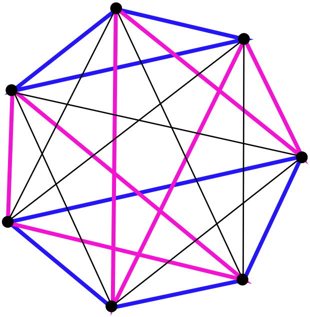

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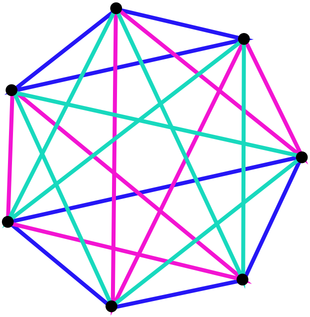

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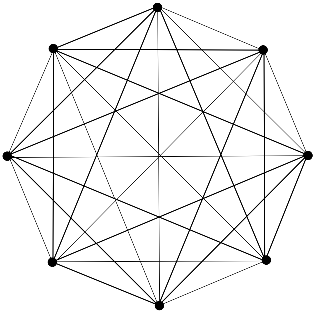
  The $n = 2k$ participants, consisting of $k$ couples, are to be seated at $t$ round tables for $k-1$ consecutive nights so that each person sits next to each other person **exactly once**, except they **never** sit next to their spouse. Can this be achieved with tables of sizes $m_1, m_2, \ldots, m_t$ assuming $m_1 + m_2 + \ldots + m_t = n$?



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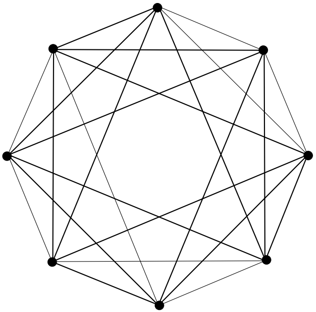

Figure: $K_8 - I$

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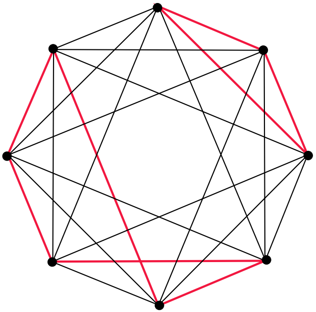

Figure: Spouse-Avoiding Variant with $n = 8$, $m_1 = 3$, and $m_2 = 5$

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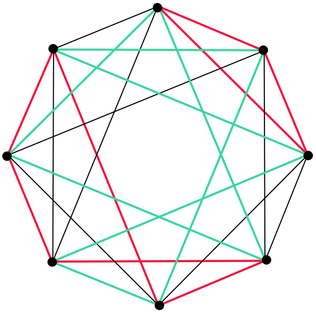

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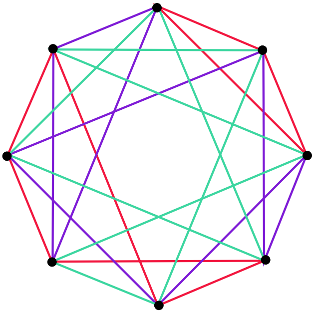

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# $OP(m_1, m_2, \ldots, m_t)$

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  assuming $n = m_1 + m_2 + \ldots + m_t$, where each $m_i \in \{3, 4, \ldots, n\}$, does there exist a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $K_n$ ($n$ odd) or $K_n - I$ ($n$ even)?

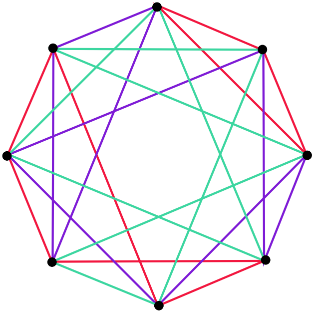

Figure: A solution to OP(3, 5)

# Oberwolfach Problem with uniform cycle length

- $OP(n; m)$:
  assuming $3 \leq m \leq n$ and $m | n$,
  does there exist a $C_m$-factorization of $K_n$ ($n$ odd) or $K_n - I$ ($n$ even)?

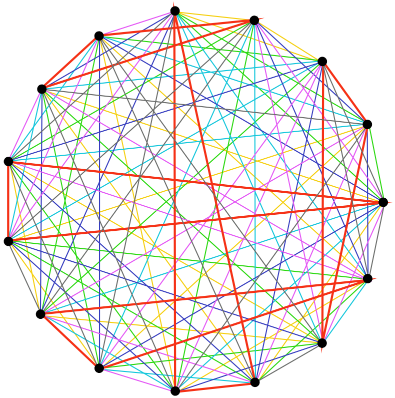

Figure: Solution to $OP(15; 3)$: a $C_3$-factorization of $K_{15}$ or KTS(15)

# OP(15;3): The Kirkman Schoolgirl Problem

- Kirkman (1850):
  "15 young ladies in a school walk 3 abreast for 7 days in succession: it is required to arrange them daily, so that no two shall walk twice abreast."
- That is, find a collection of triples from a set of 15 elements so that every pair of elements lie together in exactly one triple, and the collection of triples partitions into subsets of 5 pairwise disjoint triples.
- First solution by Cayley (1850), followed by Kirkman (1850).
- Woolhouse (1863) collects 7 non-isomorphic solutions.
- Cole (1922) proves there are precisely 7 non-isomorphic solutions.

# Oberwolfach Problem — most important results to date

- **No solution:** $OP(3,3)$, $OP(3,3,3,3)$, $OP(4,5)$, $OP(3,3,5)$
  Apart from these widely believed to be the only exceptions,
- $OP(n; m)$ **has a solution** for...
  - $m = 3$ and $n$ odd — Jiaxi, 1961-65; Ray-Chaudhuri and Wilson, 1973
  - $m = 3$ and $n$ even — Kotzig and Rosa, 1974; Baker and Wilson, 1977; Brouwer, 1978; Rees and Stinson, 1987
  - $m$ even — Alspach and Häggkvist, 1985
  - $m$ odd, $m \geq 5$, and $n \neq 4m$
    — Alspach, Schellenberg, Stinson, Wagner, 1989
  - $m$ odd, $m \geq 5$, and $n = 4m$ — Hoffman and Schellenberg, 1991
- ... and $OP(m_1, m_2, \ldots, m_t)$ **has a solution** for
  - infinitely many $n$ — Bryant and Scharaschkin, 2009
  - $n \leq 40$ — Adams, Bolstad, Bryant, Deza, Franek, Holub, Hua, Huang, Kotzig, Meszka, Rosa, 1979-2010
  - $m_1, m_2, \ldots, m_t$ all even — Bryant and Danziger, 2011
  - $t = 2$ — Traetta, 2013

# Some related results

- Oberwolfach Problem for complete multigraphs, with uniform cycle length — Gvozdjak, 1997
- Oberwolfach Problem for complete equipartite graphs and complete equipartite multigraphs, with uniform cycle length — Liu, Lick, 2003
- Oberwolfach Problem for complete equipartite graphs, with bipartite 2-factors — Bryant, Danziger, Patterson, 2015

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# Directed Oberwolfach Problem — resolvable directed cycle decompositions

- **The Directed Oberwolfach Problem:** $n$ participants are to be seated at $t$ round tables for $n-1$ consecutive nights so that each participant sits **to the right** of each other participant **exactly once**. Can this be achieved with tables of sizes $m_1, m_2, \ldots, m_t$ assuming $m_1 + m_2 + \ldots + m_t = n$?

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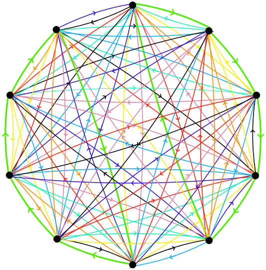

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- $OP^*(n; m)$: assuming $2 \leq m \leq n$ and $m|n$, does there exist a resolvable decomposition of $K_n^*$ into directed $m$-cycles?

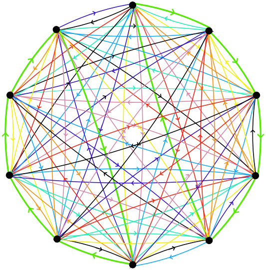

Figure: Solution to $OP^*(10; 5)$

# Results on $OP^*(n; m)$

- $OP^*(3t; 3)$ has a solution if and only if $t \neq 2$
  — Bermond, Germa, and Sotteau, 1979
- $OP^*(4t; 4)$ has a solution for all $t$ — Bennett and Zhang, 1990

**Theorem (Burgess and Šajna, 2014)**

*For $m \geq 5$: $OP^*(tm; m)$ has a solution if $m$ is even, or $t$ and $m$ are both odd.*

**Theorem (Burgess and Šajna, 2014)**

*For odd $m \geq 5$: if $OP^*(2m; m)$ has a solution, then $OP^*(tm; m)$ has a solution for all even $t$.*

**Theorem (Burgess, Francetić, Šajna, 2018)**

*$OP^*(2m; m)$ has a solution for all odd $m$, $5 \leq m \leq 49$.*

# The Spouse-Loving Variant of the Oberwolfach Problem

- **The Spouse-Loving Variant:** $n = 2k$ participants, consisting of $k$ couples, are to be seated at $t$ round tables for $k$ consecutive nights so that each person sits next to each other person **exactly once**, except they sit next to their spouse **exactly twice**. Is this possible for tables of sizes $m_1, m_2, \ldots, m_t$ if $m_1 + m_2 + \ldots + m_t = n$?

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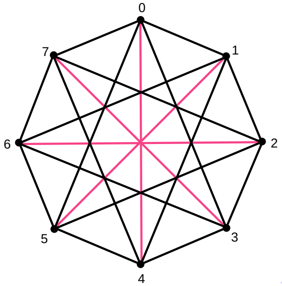
- Spouse-Avoiding Variant: $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $K_n - I$ $=$ maximum packing of $K_n$ with $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors



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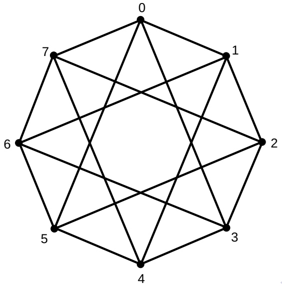

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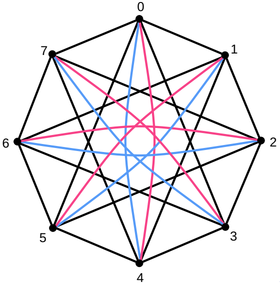
- Spouse-Loving Variant: $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $K_n + I$ = minimum covering of $K_n$ with $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factors



## Results on $\mathrm{OP}^+(m_1, m_2, \ldots, m_t)$

- $\mathrm{OP}^+(m_1, m_2, \ldots, m_t)$:
  assuming $n = m_1 + m_2 + \ldots + m_t$,
  does there exist a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $K_n + I$?
- $\mathrm{OP}^+(n; m)$:
  assuming $3 \leq m \leq n$ and $m|n$,
  does there exist a $C_m$-factorization of $K_n + I$?
- Resolvable minimum coverings by triples:
  $\mathrm{OP}^+(3t; 3)$ has a solution if and only if $t$ is even and $t \geq 6$
  — Assaf, Mendelsohn, and Stinson, 1987; Lamken and Mills, 1993

**Theorem (Bolohan, Buchanan, Burgess, Šajna, 2018$^+$)**

- *If $m_1, m_2, \ldots, m_t$ are all even, then $OP^+(m_1, m_2, \ldots, m_t)$ has a solution.*
- *If $m$ is odd, $m \geq 5$, then $OP^+(tm; m)$ has a solution for every even $t$, except possibly for $t = 4$.*

# The Honeymoon Oberwolfach Problem

- **The Honeymoon Oberwolfach Problem:** $2n$ participants, consisting of $n$ newly-wed couples, are to be seated at $t$ round tables for $2n-2$ consecutive nights so that each person sits next to each other person **exactly once**, except they sit next to their spouse **every time**. Can this be achieved with tables of sizes $m_1, m_2, \ldots, m_t$ assuming $m_1 + m_2 + \ldots + m_t = 2n$?

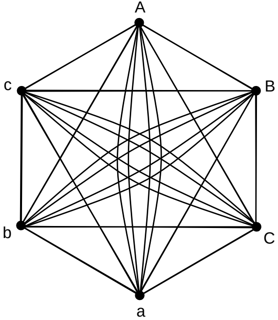



Figure: HOP with $n = 3$ and $m_1 = 6$

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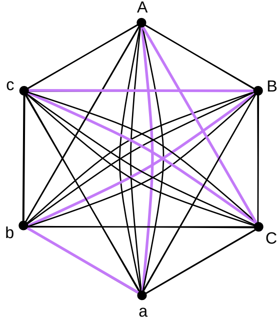


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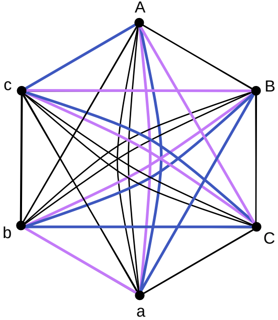


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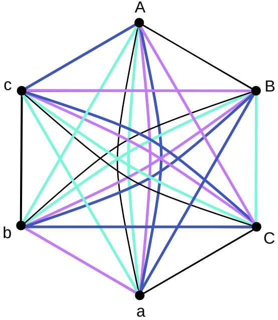


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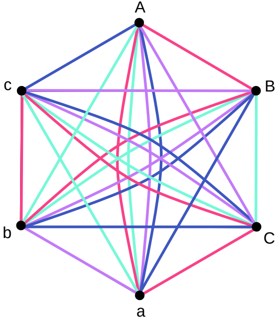


Figure: HOP with $n = 3$ and $m_1 = 6$

# HOP$(m_1, m_2, \ldots, m_t)$

- HOP$(m_1, m_2, \ldots, m_t)$:
  assuming $2n = m_1 + m_2 + \ldots + m_t$ with $m_1, m_2, \ldots, m_t$ all even, does there exist an $I$-alternating $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization of $K_{2n} + (2n-3)I$?
- HOP$(2n; 2m)$:
  assuming $2 \leq m \leq n$ and $m|n$,
  does there exist an $I$-alternating $C_{2m}$-factorization of $K_{2n} + (2n-3)I$?

- A solution to HOP$(m_1, m_2, \ldots, m_t)$ is equivalent to a semi-uniform 1-factorization of $K_{2n}$ of type $(m_1, m_2, \ldots, m_t)$

- Semi-uniform 1-factorization of type $(m_1, m_2, \ldots, m_t)$:
  1-factorization $\{F_0, F_1, \ldots, F_{r-1}\}$ such that for all $i \neq 1$,
  $F_0 \cup F_i$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor

# Special 1-factorizations

A 1-factorization $\{F_0, F_1, \ldots, F_{r-1}\}$ is called...

- Semi-uniform of type $(m_1, m_2, \ldots, m_t)$ if
  $F_0 \cup F_i$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor for all $i \neq 0$
- Semi-perfect :
  $F_0 \cup F_i$ is a Hamilton cycle for all $i \neq 0$
- Uniform of type $(m_1, m_2, \ldots, m_t)$ if
  $F_i \cup F_j$ is a $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factor for all $i \neq j$
- Perfect if
  $F_i \cup F_j$ is a Hamilton cycle for all $i \neq j$
- Sequentially uniform if
  it admits a cyclic ordering $(F_0, F_1, \ldots, F_{r-1})$ such that the 2-factors $F_i \cup F_{i+1}$ are pairwise isomorphic for all $i \in \mathbb{Z}_r$
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# Known results on special 1-factorizations

- Kotzig's Conjecture (1964):
  $K_{2n}$ admits a perfect 1-factorization for all $n$
  - Confirmed for many $n$, open in general
- Královič and Královič, 2005:
  $K_{2n}$ admits a semi-perfect 1-factorization for all $n$
- Dinitz, Dukes, Stinson, 2005:
  $K_{2n}$ admits a sequentially perfect 1-factorization for all $n$

# Results on HOP$(m_1, m_2, \ldots, m_t)$

**Theorem (Burgess, Lepine, Šajna, 2018$^+$)**

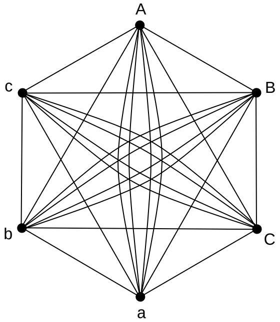
*Assume* $2 \leq m_1 \leq m_2 \leq \ldots \leq m_t$ *and* $n = m_1 + m_2 + \ldots + m_t$.
*Then* HOP$(2m_1, 2m_2, \ldots, 2m_t)$ *has a solution if*

1. $n$ *is odd and* OP$(m_1, m_2, \ldots, m_t)$ *has a solution; or*
2. $m_i \equiv 0 \pmod 4$ *for all* $i$*; or*
3. $n$ *is odd and* $t = 2$*; or*
4. $n$ *is odd,* $n < 40$*, and* $m_1 \geq 3$*; or*
5. $n \leq 9$.

**Theorem (BLŠ, 2018$^+$)**

*Assume* $2 \leq m \leq n$.
*Then* HOP$(2n; 2m)$ *has a solution if and only if* $n \equiv 0 \pmod m$.

# Modelling a table



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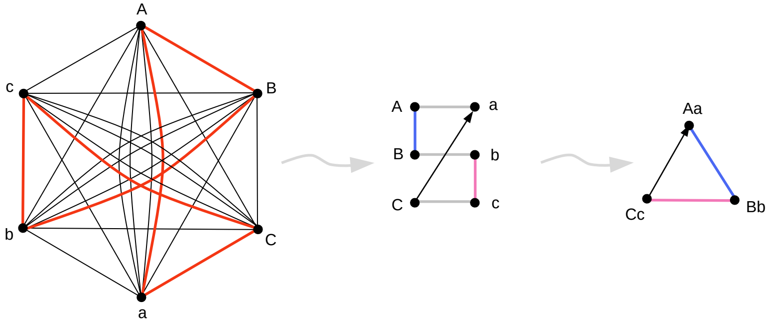

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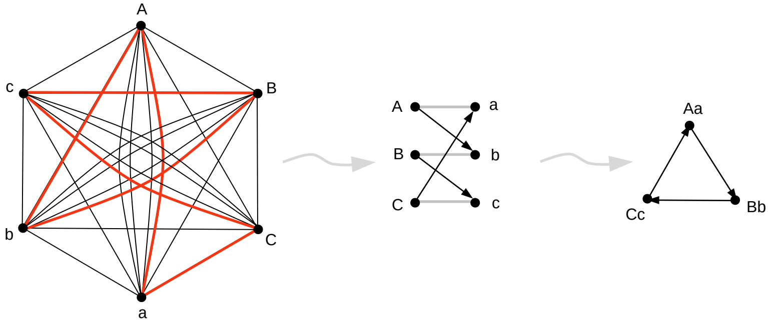
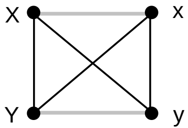


# From $K_{2n} + (2n-3)I$ to a colour-oriented $4K_n$



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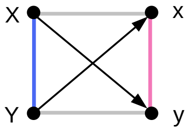

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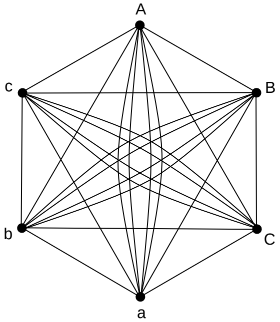

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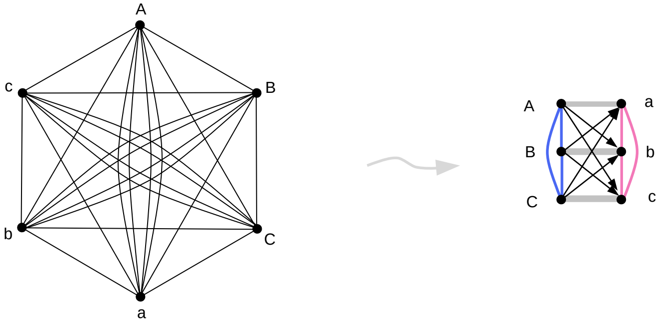

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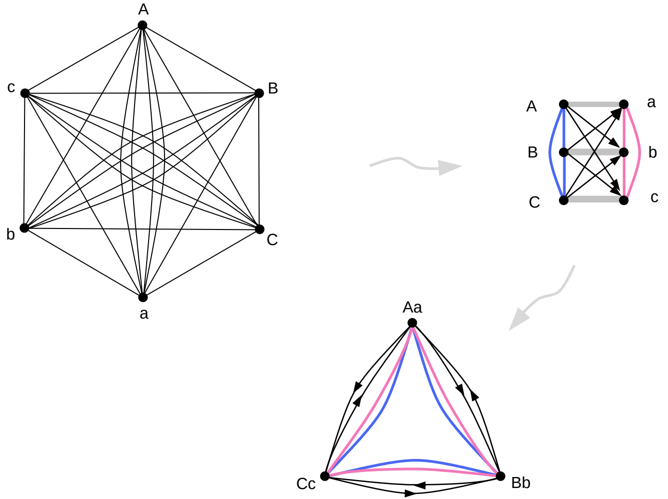
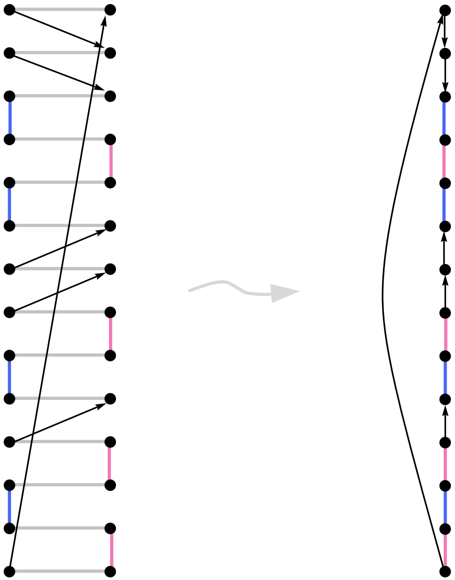


# Admissible cycles in a colour-oriented $4K_n$



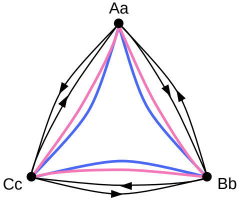
# General approach: HOP colouring-orientation

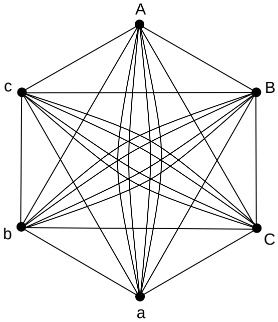
- HOP colouring-orientation of $4G$:
  a 3-edge colouring of $4G$ (with colours pink, blue, and black), and an orientation of the black edges such that each 4-set of parallel edges contains one pink edge, one blue edge, and two opposite black arcs
- $4G^{\bullet}$: $4G$ with a given HOP-colouring-orientation
- HOP 2-factorization of $4G$:
  in each cycle, any two adjacent edges satisfy one of:
  - one is blue, one pink; or
  - both are black and directed in the same way;
  - one is blue, one black, directed towards the blue edge; or
  - one is pink, one black, directed away from the pink edge.

**Theorem (BLŠ, 2018$^{+}$)**

*Let $n = m_1 + m_2 + \ldots + m_t$. HOP$(2m_1, 2m_2, \ldots, 2m_t)$ has a solution if and only if $4K_n^{\bullet}$ admits an HOP $(C_{m_1}, C_{m_2}, \ldots, C_{m_t})$-factorization.*

# Solution to HOP(6; 6)



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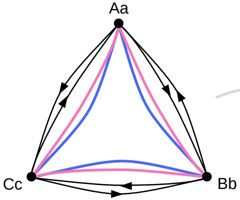


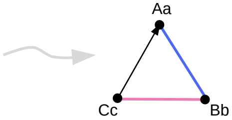


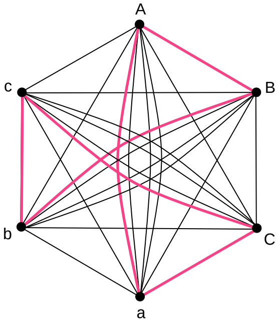

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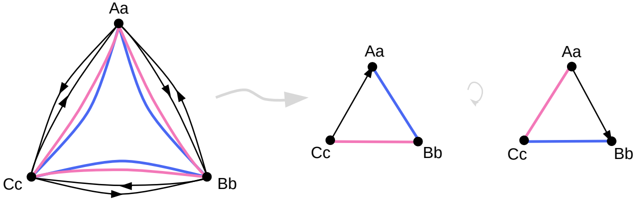


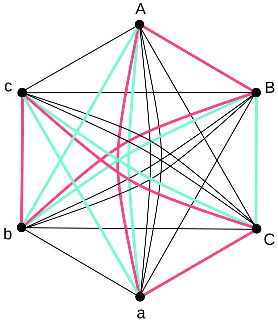

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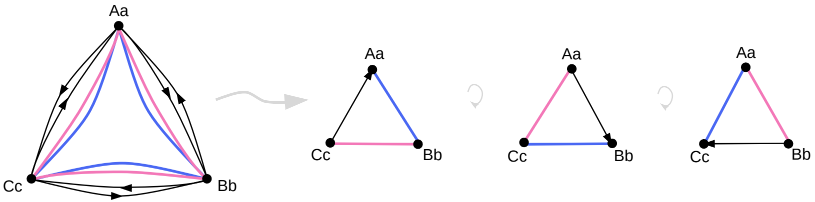


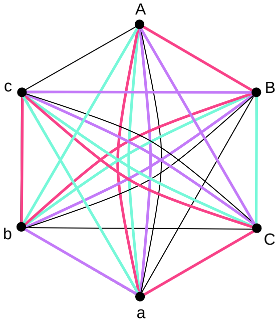

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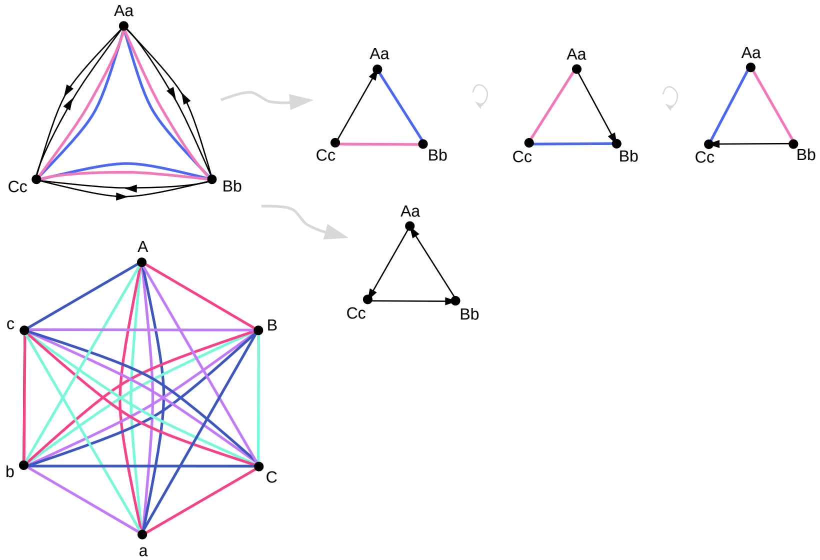


# Lemma 1

**Lemma**

*Let* $G = H_1 \oplus \ldots \oplus H_s$.

1. *If* $H_1, \ldots, H_s$ *are spanning, and each* $4H_i^\bullet$ *admits an HOP* $(C_{m_1}, \ldots, C_{m_t})$*-factorization, then* $4G^\bullet$ *admits an HOP* $(C_{m_1}, \ldots, C_{m_t})$*-factorization.*
2. *If* $H_1, \ldots, H_s$ *are* $r$*-regular, pairwise vertex-disjoint, and each* $4H_i^\bullet$ *admits an HOP* $C_m$*-factorization, then* $4G^\bullet$ *admits an HOP* $C_m$*-factorization*

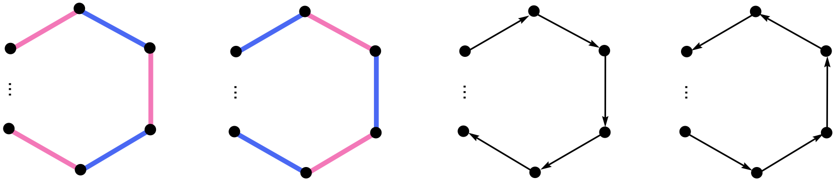
# Lemma 2

**Lemma**

*If $G$ admits a $(C_{m_1}, \ldots, C_{m_t})$-factorization, then $4G^{\bullet}$ admits an HOP $(C_{m_1}, \ldots, C_{m_t})$-factorization.*

Proof. By Lemma 1, it suffices to prove that if $C$ is an $m$-cycle, then $4C^{\bullet}$ admits an HOP $C_m$-factorization.

*Case 1:* $m$ is even.



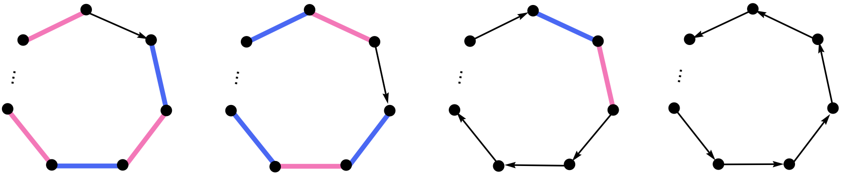
# Lemma 2

**Lemma**

*If $G$ admits a $(C_{m_1}, \ldots, C_{m_t})$-factorization, then $4G^\bullet$ admits an HOP $(C_{m_1}, \ldots, C_{m_t})$-factorization.*

PROOF. By Lemma 1, it suffices to prove that if $C$ is an $m$-cycle, then $4C^\bullet$ admits an HOP $C_m$-factorization.

*Case 2:* $m$ is odd.



■

# From OP to HOP

**Corollary**

*Assume* $2 \leq m_1 \leq m_2 \leq \ldots \leq m_t$, $n = m_1 + m_2 + \ldots + m_t$ *is odd, and* $OP(m_1, m_2, \ldots, m_t)$ *has a solution.*
*Then* $HOP(2m_1, 2m_2, \ldots, 2m_t)$ *has a solution.*

PROOF.

- Since $n$ is odd and OP$(m_1, \ldots, m_t)$ has a solution, $K_n$ admits a $(C_{m_1}, \ldots, C_{m_t})$-factorization.
- Hence $4K_n^\bullet$ admits an HOP $(C_{m_1}, \ldots, C_{m_t})$-factorization by Lemma 2.
- Hence HOP$(2m_1, 2m_2, \ldots, 2m_t)$ has a solution.

∎

# HOP with uniform cycle lengths

**Theorem (BLŠ, $2018^+$)**

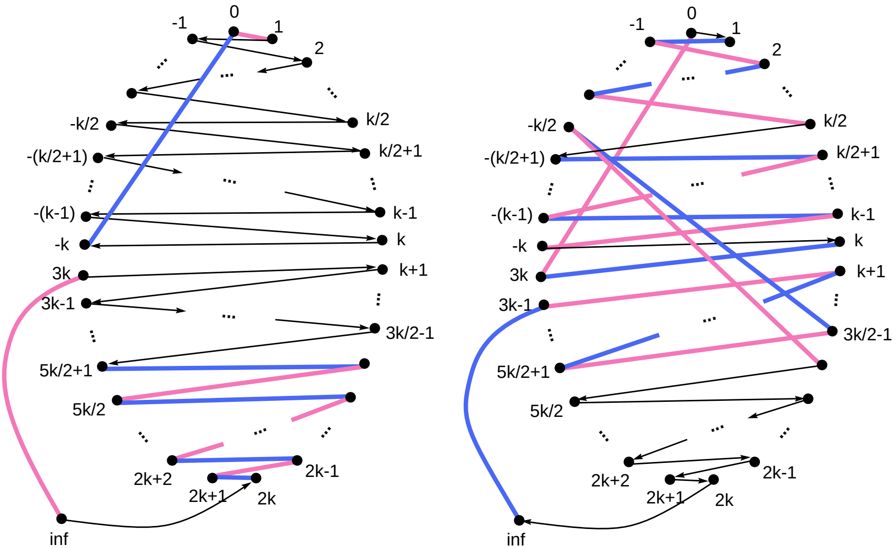
*Assume $2 \leq m \leq n$.*
*Then $HOP(2n; 2m)$ has a solution if and only if $n \equiv 0 \pmod{m}$.*

PROOF. Necessity is clear. We prove sufficiency for odd $m \geq 5$ only.

- Assume $n \equiv 0 \pmod{m}$.
- If $n$ is odd, then $\mathrm{OP}(n; m)$ has a solution, so $\mathrm{HOP}(2n; 2m)$ has a solution.
- Hence assume $n$ is even.
- Suppose first that $4K_{2m}^{\bullet}$ and $4K_{4m}^{\bullet}$ both admit HOP $C_m$-factorizations.
- Let $t = \frac{n}{2m}$, and assume $t \geq 3$.
- Decompose $K_n = t \cdot K_{2m} \oplus K_{t[2m]}$.
- $4K_{2m}^{\bullet}$ admits an HOP $C_m$-factorization by supposition.
- $K_{t[2m]}$ admits a $C_m$-factorization by [Liu, 2003].
- Hence $4K_{t[2m]}^{\bullet}$ admits an HOP $C_m$-factorization by Lemma 2.
- Thus $4K_n^{\bullet}$ admits an HOP $C_m$-factorization by Lemma 1.

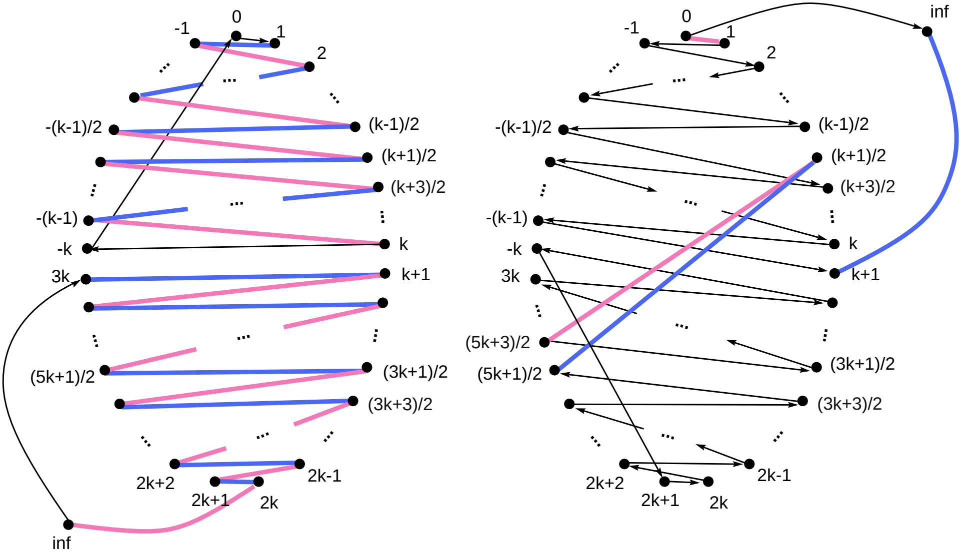
# HOP $C_m$-factorization of $4K^{\bullet}_{2m}$ for odd $m \geq 5$

Starter $C_m$-factors for $m = 2k + 1$ with $k$ even:



# HOP $C_m$-factorization of $4K_{2m}^{\bullet}$ for odd $m \geq 5$

Starter $C_m$-factors for $m = 2k+1$ with $k$ odd:



# HOP $C_m$-factorization of $4K^{\bullet}_{4m}$ for odd $m \geq 5$

- From the $C_m$-factorization of $2K_{4m}$ by [Gvozdjak, 1997], we obtain an HOP $C_m$-factorization of $4K^{\bullet}_{4m}$.
- Example: $m = 5$

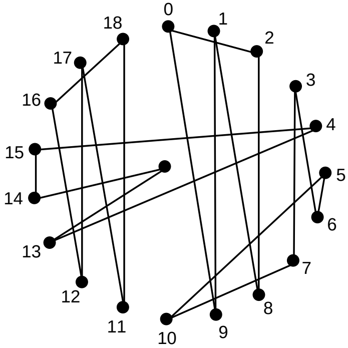



Figure: Starter $C_m$-factor for $2K_{4m}$

# HOP $C_m$-factorization of $4K^{\bullet}_{4m}$ for odd $m \geq 5$

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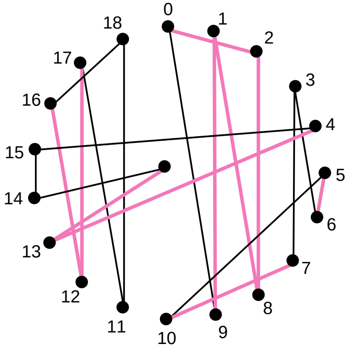


Figure: Colour one edge of each difference pink so that each cycle has an even number of pink edges

# HOP $C_m$-factorization of $4K^{\bullet}_{4m}$ for odd $m \geq 5$

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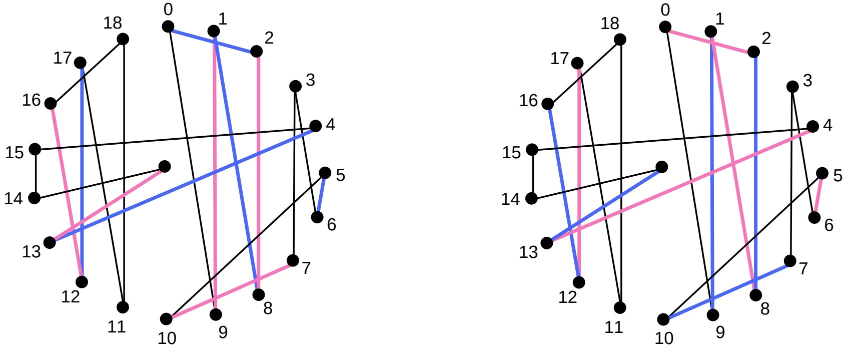


Figure: Re-colour every other pink edge blue in two different ways

# HOP $C_m$-factorization of $4K^\bullet_{4m}$ for odd $m \geq 5$

- From the $C_m$-factorization of $2K_{4m}$ by [Gvozdjak, 1997], we obtain an HOP $C_m$-factorization of $4K^\bullet_{4m}$.
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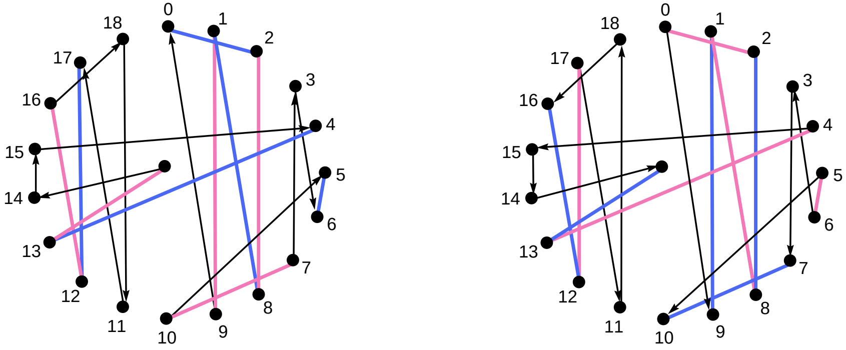


Figure: Black edges can now be oriented appropriately to yield starter $C_m$-factors for an HOP $C_m$-factorization of $4K^\bullet_{4m}$

# HOP with uniform cycle lengths — conclusion

**Theorem (BLŠ, 2018$^+$)**

*Assume* $2 \le m \le n$.
*Then* $HOP(2n; 2m)$ *has a solution if and only if* $n \equiv 0 \pmod{m}$.

PROOF. (continued — for odd $m \ge 5$)

- So $4K^{\bullet}_{2m}$ and $4K^{\bullet}_{4m}$ both admit an HOP $C_m$-factorization.
- Hence $4K^{\bullet}_{n}$ admits an HOP $C_m$-factorization for all $n \equiv 0 \pmod{m}$.
- Therefore $\mathrm{HOP}(2n; 2m)$ has a solution.

■

# HOP conclusion

## Theorem (BLŠ, 2018$^+$)

*Assume* $2 \leq m_1 \leq m_2 \leq \ldots \leq m_t$ *and* $n = m_1 + m_2 + \ldots + m_t$*.*
*Then* $HOP(2m_1, 2m_2, \ldots, 2m_t)$ *has a solution if*

1. $n$ *is odd and* $OP(m_1, m_2, \ldots, m_t)$ *has a solution; or*
2. $m_i \equiv 0 \pmod 4$ *for all* $i$*; or*
3. $n$ *is odd and* $t = 2$*; or*
4. $n$ *is odd,* $n < 40$*, and* $m_1 \geq 3$*; or*
5. $n \leq 9$.

## Conjecture

*The obvious necessary conditions for* $HOP(2m_1, \ldots, 2m_t)$ *to have a solution — or equivalently,*
*for* $K_{2n}$ *to admit a semi-uniform 1-factorization of type* $(2m_1, \ldots, 2m_t)$ *— are also sufficient.*

# Thank you!