Two problems about symmetries of finite graphs

Presented by

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Based on joint work with

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and 65th birthday of Dragan Marušič

1st problem

CONTROLLING THE AUTOMORPHISM GROUP

OF A COVERING GRAPH

Motivation, part 1

- Let Γ be a finite connected cubic G-arc-transitive graph. Then G is of one of 7 "types":
  - G is 1-arc-regular;
  - G is 2-arc-regular (two "types");
  - G is 3-arc-regular;
  - G is 4-arc-regular (two "types");
  - G is 5-arc-regular.
- It is easy to construct pairs (Γ, G) for each of the above possibilities.
- Problem (Djoković and Miller, 1980): Can this be achieved with $G = \mathrm{Aut}(\Gamma)$?

Motivation, part 2

- Let Γ be a finite connected tetravalent G-half-arc-transitive graph. Then (by Marušič and Nedela):
  - $|G_v| = 2^s$ for some $s \geq 1$;
  - for every s, there is a finite number of "types" for G;
- Easy to find pairs (Γ, G) for each of the above types.
- Marušič, Nedela, 2001: Can this be achieved with $G = \mathrm{Aut}(\Gamma)$?
- Yes, for some types, unknown in general!

Possible general approach to such problems

General problem: We are given a pair (Γ, G) of a given "type", but such that $G < \mathrm{Aut}(\Gamma)$. Can we find another pair $(\tilde{\Gamma}, \tilde{G})$ of the same "type", where $\tilde{G} = \mathrm{Aut}(\tilde{\Gamma})$.

Covering projections, part I

Let $\tilde{\Gamma}$ and Γ be connected graphs.

A graph morphism $\wp\colon \tilde{\Gamma} \to \Gamma$ is a covering projection provided that

- $\wp$ is a surjection (epimorphism);
- for every $v \in V_{\tilde{\Gamma}}$ the restriction $\wp_v\colon \tilde{\Gamma}(v) \to \Gamma(\wp(v))$ is a bijection. The valence is preserved.

Fibres and induced automorphisms

Let $\wp\colon \tilde{\Gamma} \to \Gamma$ be a covering projection.

- For a *vertex* v of Γ, the preimage $\wp^{-1}(x)$ is called a fibre of v.

- An automorphism $\tilde{g} \in \mathrm{Aut}(\tilde{\Gamma})$ that maps fibres to fibres induces an automorphism g of Γ.
- In this case we say: $\tilde{g}$ projects, g lifts, and $\tilde{g}$ is a lift of g.
- Let $G \leq \mathrm{Aut}(\Gamma)$. If every $g \in G$ lifts, then G lifts. The set $\tilde{G}$ of all lifts of all $g \in G$ is a group, called the lift of G.
- The lift of the trivial group $\langle \mathrm{id}_\Gamma \rangle \leq \mathrm{Aut}(\Gamma)$ is called the group of covering tranformations ... $\mathrm{CT}(\wp)$.

Regular covers and its nice feature

If $\mathrm{CT}(\wp)$ is transitive on every fibre, then $\wp$ is a regular covering projection.

Let $\wp\colon \tilde{\Gamma} \to \Gamma$ be a regular covering projection. Suppose that $G \leq \mathrm{Aut}(\Gamma)$ lifts to $\tilde{G}$. Then:

- G is vertex-transitive iff $\tilde{G}$ is vertex-transitive;
- G is edge-transitive iff $\tilde{G}$ is edge-transitive;
- G is s-arc–transitive iff $\tilde{G}$ is s-arc-transitive;

In short, regular covering projections preserve "type".

The problem

Recall our problem: For a (Γ, G) of a given "type", find another pair $(\tilde{\Gamma}, \tilde{G})$ of the same "type" satisfying $\tilde{G} = \mathrm{Aut}(\tilde{\Gamma})$.

We can now solve this by:

finding a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ s.t.:

1. G lifts along $\wp$, but no larger group does;
2. Every automorphism of $\mathrm{Aut}(\tilde{\Gamma})$ projects to some automorphism of Γ.

This works since "type" is preserved by $\wp$.

Main result

Theorem (P. Spiga, PP, 2017)

Let Γ be a finite graph s.t. $\mathrm{Aut}(\Gamma)$ acts faithfully on the integral cycle space $H_1(\Gamma, \mathbb{Z})$, let $G \leq \mathrm{Aut}(\Gamma)$ and let p be an odd prime. Then there exists a regular covering projection $\wp \colon \tilde{\Gamma} \to \Gamma$ s.t.

- *G is the maximal group that lifts along $\wp$;*
- *$\mathrm{CT}(\wp)$ is a (finite) p-group.*

We are not quite happy with this. We would like to add:

- Every automorphism of $\tilde{\Gamma}$ projects to an automorphism of Γ.

We conjecture this is true, but we have no proof!

Some consequences: cubic arc-transitive

Nevertheless, in some cases this theorem yields the desired result.

For example:

Theorem

Let Γ be a finite cubic G-arc-transitive graph. Then there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ (with $\tilde{\Gamma}$ finite) such that $\mathrm{Aut}(\tilde{\Gamma})$ is the lift of G.

Some consequences, 2-arc-transitive

Theorem

Let Γ be a finite $(G,2)$-arc-transitive graph (or G-arc-transitive of prime valence). Then there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ (with $\tilde{\Gamma}$ finite) such that $\mathrm{Aut}(\tilde{\Gamma})$ is the lift of G.

... and several other similar theorems...

Alas

Our theorem is not good enough to solve the problem of Marušič and Nedela:

Does there exist a tetravalent half-arc-transitive graph of every possible "type" (in particular, with arbitrary large non-abelian vertex-stabiliser).

But if "conjecture" is true, then the answer to the above is affirmative.

2nd problem

FIXICITY OF GRAPHS

Motion and Fixicity

Let G be a permutation group on Ω and $g \in G$.

Support of g: $\mathrm{Supp}(g) := \{\omega \in \Omega \colon \omega^g \neq \omega\}$

Fixed points of g: $\mathrm{Fix}(g) := \{\omega \in \Omega \colon \omega^g = \omega\}$

Motion of G: $\mathrm{mt}(G) := \min\{|\mathrm{Supp}(g)| : g \in G, g \neq 1\}$

Fixicity of G: $\mathrm{fx}(G) := \max\{|\mathrm{Fix}(g)| : g \in G, g \neq 1\}$

Let Γ be a graph with $\mathrm{Aut}(\Gamma)$ acting on $V(\Gamma)$.

Motion of Γ: $\mathrm{mt}(\Gamma) := \mathrm{mt}(\mathrm{Aut}(\Gamma))$

Fixicity of G: $\mathrm{fx}(\Gamma) := \mathrm{fx}(\mathrm{Aut}(\Gamma))$

Fixicity of cubic vertex-transitive graphs

Question: Can we somehow non-trivially bound the fixicity?

Problem: Suppose we are given a class of graphs $\mathcal{G}$. Find a a function $f\colon \mathbb{N} \to \mathbb{N}$ (as slowly growing as possible) such that all (but finitely many) graphs $\Gamma \in \mathcal{G}$ satisfy

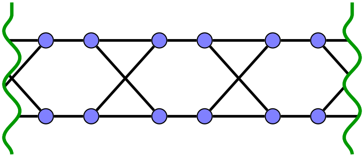
$$\mathrm{fx}(\Gamma) \leq f(|\mathrm{V}(\Gamma)|)$$

We will consider this question for classes $\mathcal{G}$ of

- cubic vertex-transitive graphs;
- cubic arc-transitive graphs.

Split Praeger-Xu graphs

Some cubic vertex-transitive graphs have very large fixicity:



Split wreath graph SW_m : $\mathrm{fx}(SW_m) = n - 4$

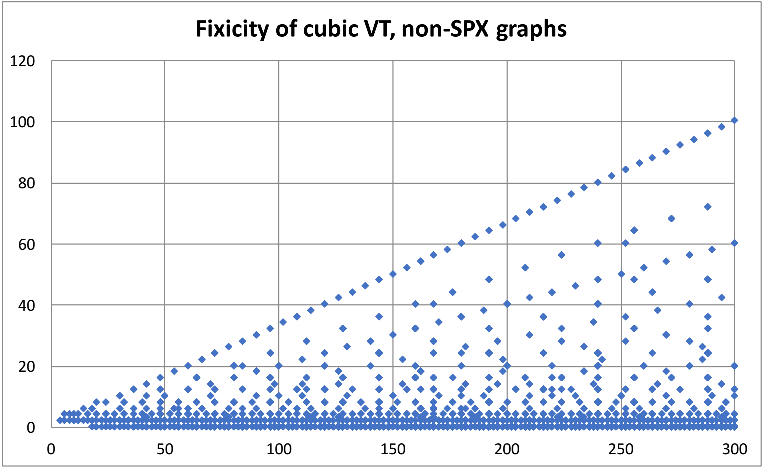
More generally, split Praeger-Xu graphs $SPX(n, k)$ satisfy

$$\mathrm{fx}(SPX(m, k)) = n - k2^{k+1}$$

Here

$$n = |\mathrm{V}(SPX(m, k))| = m2^k$$

Fixicity of cubic vertex-transitive graphs



$$f(n) = \frac{1}{3}n$$

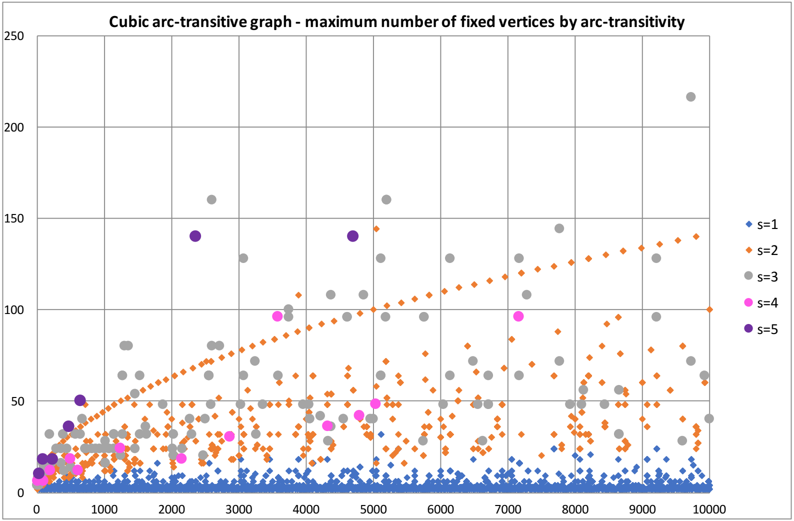
Fixicity of cubic vertex-transitive graphs – conjecture

Conjecture

If Γ is a finite connected cubic vertex-transitive graph, then either it is isomorphic to a SPX-graph or

$$\mathrm{fx}(\Gamma) \leq \frac{1}{3} n.$$

Fixicity of cubic arc-transitive graphs



$$f(n) = \sqrt{2n}$$

Cubic arc-transitive graphs with fixicity $\sqrt{2n}$

$G = \langle u, v, t \mid u^m, v^m, t^2, [u, v], u^t = u^{-1}, v^t = v^{-1} \rangle \cong \mathbb{Z}_m^2 : \mathbb{Z}_2$

$a = ut,\ b = vt,\ c = u^{-1}v^{-1}t$ (three involutions)

$\Gamma = \mathrm{Cay}(G; \{a, b, c\})$

$\sigma\colon u \mapsto v \mapsto u^{-1}v^{-1},\ t \mapsto t$

$\sigma \in \mathrm{Aut}(G),\ a \mapsto b \mapsto c \mapsto a \quad \sigma \in Aut(\Gamma)_{1_G}$

Suppose there exists $\lambda \in \mathbb{Z}_m^*$ such that $\lambda^2 + \lambda + 1 = 0$.

Then σ fixes pointwise $\langle u^{-1}v^{\lambda}, t\rangle \cong \mathbb{Z}_m : \mathbb{Z}_2$, hence:

$$\mathrm{fx}(\Gamma) \geq 2m = \sqrt{2n}$$

Fixicity of cubic arc-transitive graphs

Recall the problem: Find a function $f : \mathbb{N} \to \mathbb{N}$ (as slowly growing as possible) such that all (but finitely many) cubic arc-transitive graphs Γ satisfy

$$\mathrm{fx}(\Gamma) \leq f(|\mathrm{V}(\Gamma)|)$$

By previous result, $f(n)$ has to grow at least as fast as $\sqrt{2n}$.

Theorem (Spiga, Lehner, PP)

For every positive constant α all but finitely many connected cubic arc-transitive graphs satisfy

$$\mathrm{fx}(\Gamma) < \alpha|\mathrm{V}(\Gamma)|.$$

This shows that the "optimal" $f(n)$ is sublinear, but at least $\sqrt{2n}$.