Graphs with small distinguishing index

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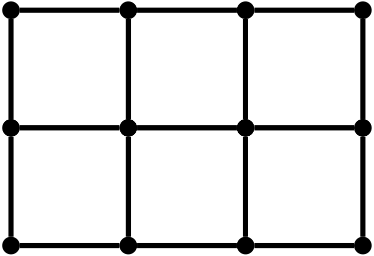
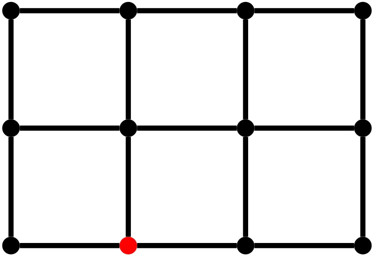
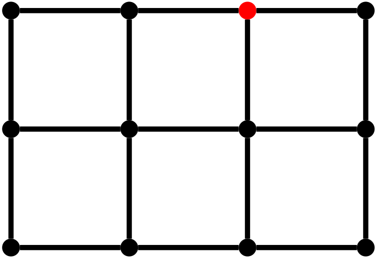
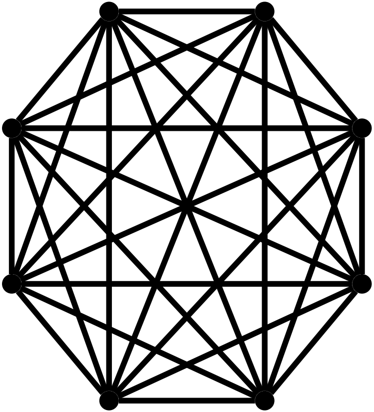






Example



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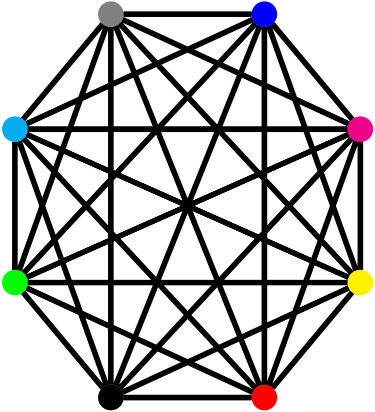

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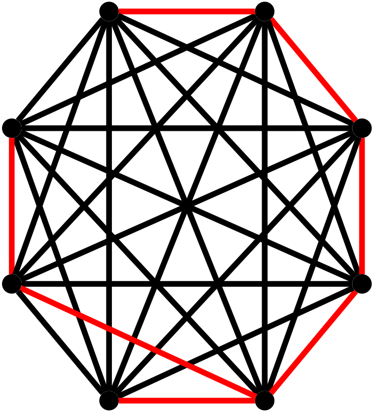

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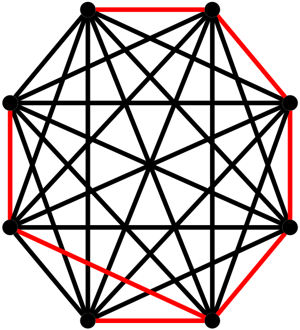

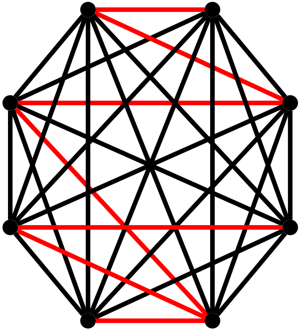

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- Assumption: $|G| \geq 3$

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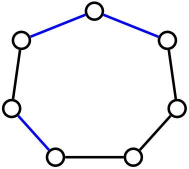

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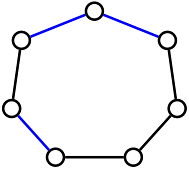

$D'(K_n) = 3, n = 3, 4, 5, \quad D'(K_n) = 2, n \geq 6$.

Trees

Def. A tree T is bisymmetric (resp. symmetric) if it has a central edge e_c (resp. a central vertex v_c), all leaves are at the same distance from e_c (resp. v_c) and every vertex that is not a leaf has the same degree.

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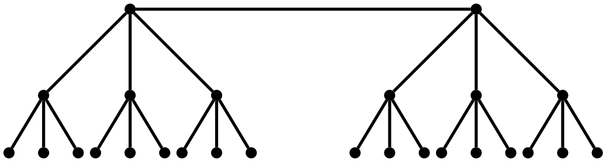

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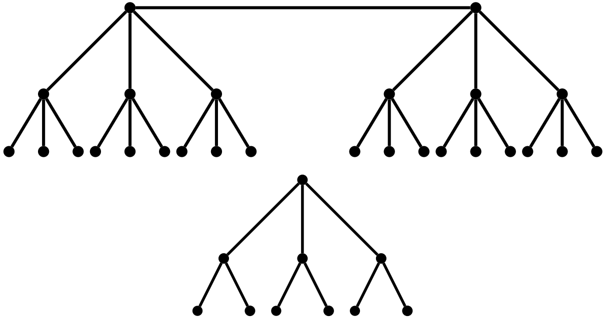

General bounds

Thm. (*Kalinowski & P. 2015*)
If T is a tree of order $n \geq 3$, then

$$D'(T) \leq \Delta(T).$$

Moreover, equality is achieved if and only if T is either a symmetric or a bisymmetric tree.

- Kalinowski, Pilśniak, *Distinguishing graphs by edge colourings*, European J. Combin. 45 (2015)

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Thm. (*Kalinowski & P. 2015*) If G is a connected graph of order $n \geq 3$, then

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except for three small cycles C_3, C_4 or C_5.

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Graphs with $D'(G) = \Delta(G)$

- Pilśniak, *Improving upper bounds for the distinguishing index*, Ars Math. Contemp. 13 (2017)

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Thm. (*P. 2017*)
Let G be a connected graph with $\Delta(G) \geq 3$. If G is neither a symmetric nor a bisymmetric tree, then

$$D'(G) \leq \Delta(G) - 1$$

unless G is K_4 or $K_{3,3}$.

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$D'(G) = \Delta(G) + 1$ iff $G \in \{C_3, C_4, C_5\}$,
$D'(G) = \Delta(G)$ iff $G \in \{K_4, K_{3,3}\} \cup \{C_n : n \geq 6\}$,
or G is either a symmetric or a bisymmetric tree.

Key lemma

Def. A graph G is almost spanned by a subgraph H if H is a spanning subgraph of $G - v$ for some $v \in V(G)$.

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Key lemma

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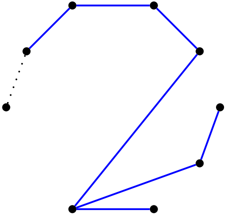

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G contains an asymmetric spanning or almost spanning subgraph.

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Thm. (*Barnette 1966*) Every 3-connected planar graph has a spanning tree T with $\Delta(T) \leq 3$.

Thm. (*P. & Tucker 2018+*) If G is a 3-connected planar graph different from K_4, then

$$D'(G) \leq 2.$$

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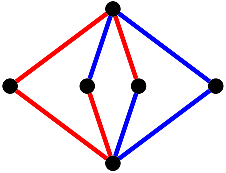

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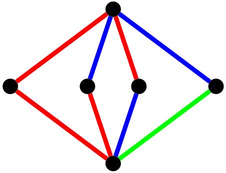

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Thm. (*Kargul, Musial & Pal, 2018+*) If G is a connected claw-free graph and $|G| \geq 7$, then

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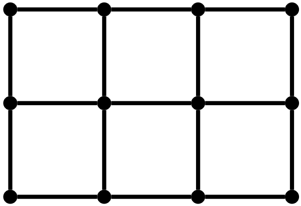
vertex set: $V(G) \times V(H)$

Ex.  $P_4 \square P_3$

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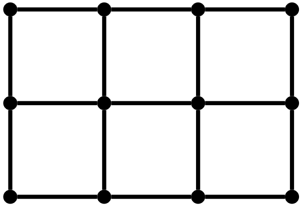
  $E(G \square H) \; = \; \{(x,u)(y,v) \mid (xy \in E(G) \wedge u = v) \vee (x = y \wedge uv \in E(H))\}$

Ex.  $P_4 \square P_3$

The Cartesian power of a graph

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If G is a connected graph of order $n \geq 3$, then $D'(G^k) = 2$.

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- Since a hypercube is Hamiltonian for $k \geq 3$.

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The Cartesian product of countable graphs

Thm. (*Broere & P. 2017*) If G is a connected prime, countably infinite graphs, then $D'(G^k) = 2$, for any $k \geq 2$.

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Thm. (*Broere & P., 2017*) $D'(K_2^{\aleph_0}) = 2$.

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Total colourings

$D''(G)$ - total distinguishing number

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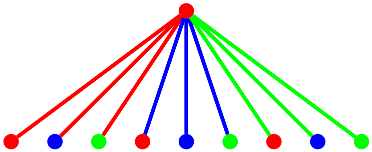

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Thm. *(Kalinowski, P. & Woźniak 2016)* If G is a connected graph, then

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Thm. *(Kalinowski, P. & Woźniak 2016)* If G is a connected graph, then

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Moreover, if $\chi''(G) \geq \Delta(G) + 2$, then

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THANK YOU VERY MUCH!