Cayley graphs, integer flows

CQ Zhang
(The # 3 in Brian's Family)

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79th birthday: 2017-05-29, in Sanya, China

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2017-05-29, with 2nd & 3rd generations

2017-05-29, with 2nd & 3rd generations

2018-05-28, with 2nd & 3rd generation

80th birthday: 2018-05-29, with 2nd generation

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Introduction of flow theory

k-face-color

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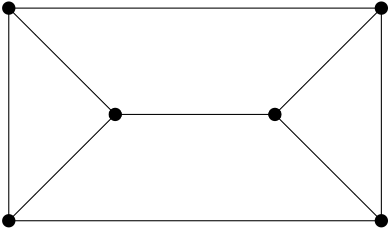

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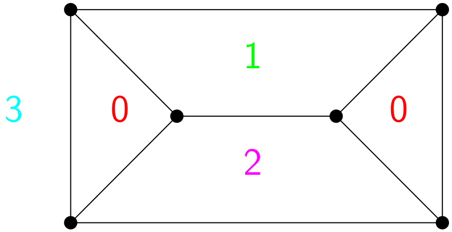

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- (1) $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$;
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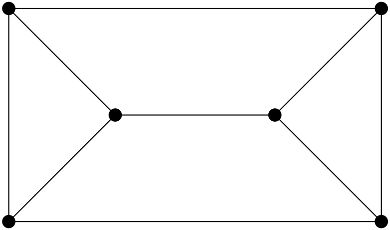

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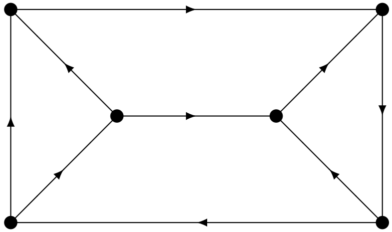

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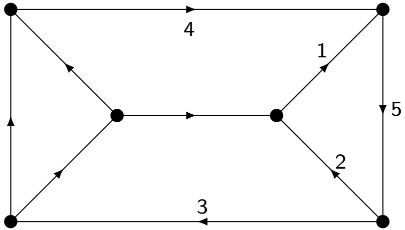

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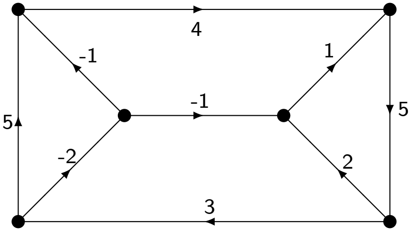

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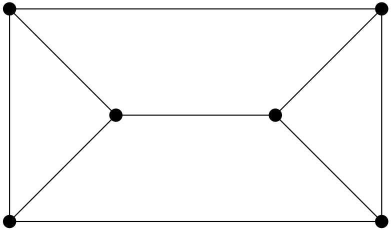

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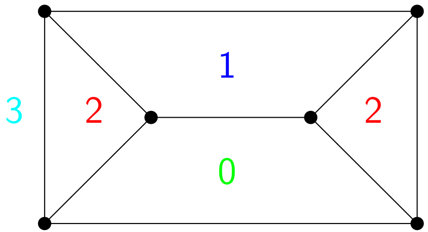

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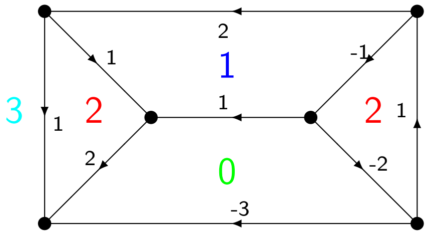

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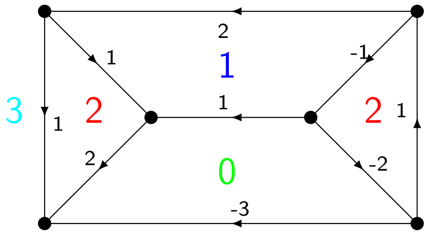

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Robertson-Sanders-Seymour-Thomas: cubic, P_{10}-minor free $\Rightarrow$ 4-NZF;

Flow for Cayley graphs

Some equivalent properties of integer flows

and

how to find nowhere-zero k-flows

and

some recent discoveries

4-flow

3-edge-coloring

Fact (Tutte). Equivalent if G is cubic.
* 4-flow
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For Cayley Graphs:
We only need to work on cubic graphs.

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$\Rightarrow G$ has a 4-NZF $f = f_4 + f_2$.

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$f = f_2 + f_4$ is 4-NZF of G.

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3-flow

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Open problem. Tutte's Conjecture holds for Cayley graphs. (That is, every t-regular Cayley graph with $t \geq 5$ has 3-NZF.)

3-NZF and Modulo 3-orientation

Definition. Let $k \in Z^+$.
An orientation D of G is a **modulo k-orientation** if

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Theorem (Tutte). Equivalent statements,
(1) G admits a nowhere-zero 3-flow
(2) G admits a modulo 3-orientation.

Cayley graphs with 3-flow

Theorem. (Primož Potočnik , Martin Škoviera , Riste Škrekovski , 2005 DM) Tutte's 3-flow conjecture is true for abelian groups.

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Theorem. (Li and Li 2015 FMC) Tutte's 3-flow conjecture is true for
(1) generalized dihedral groups and
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5-flow

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Again, we only need to pay attention to cubic case.

Circular flow, and, flow index

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Definition Let $r \in Q^+$, and (D, f) be a flow of G.
If $f : E(G) \to [1, r-1]$,
then (D, f) is a circular r-flow.

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Definition The flow index of G is

$$\phi(G) = \min\{r \in R^+ : \quad G \text{ admits a nowhere-zero circular } r\text{-flow}\}.$$

Tutte's Theorem and one of our new results

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(That is, $\phi(G) < 3$)
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G has 3 edge-disjoint spanning trees $\Rightarrow \phi(G) \leq 3$.

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Note, for Cayley graphs,

$$\text{edge-connectivity} = \text{degree}$$

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* 5-regular $\Rightarrow$ $\phi \leq 3$?

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* 3-regular $\Rightarrow$ $\phi < 5$?
(Weaker versions of ALZ-conjecture)

Summary of open problems

flow index ((for Cayley graphs)

	2	<3	3	<4	4	<5	5	<6	6 (Seymour)
Even	yes								yes
3-regular	no			no	open			open.	yes
5-regular	no	open		open.	yes		Jaeger		yes
7-regular	no	open.	yes	Lovasz-Thomassen-Wu-Z					yes
9-regular	no	yes		Li-Thomassen-Wu-Z					yes

Note, the edge connectivity of a Cayley graph = it is degree.

Happy birthdays to

Brian

and

Dragan