# Connected Sets and Connected Partitions of a Graph

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Graphs, Groups, and More
UP FAMNIT, Koper, Slovenia, 2018

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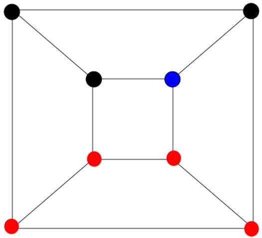
# Graph Analogs

Simple graph $G$ with vertex set $V = \{1, 2, \dots, n\}$

$C(G) =$ the number of connected subsets of $V$

$P(G) =$ the number of connected partitions of $V$



# Connected Subsets of $V$

$C(G) =$ the number of connected subsets of $V$

For any connected graph $G$ with $n$ vertices,

$$C(G) \leq C(K_n) = 2^n - 1$$

$$C(G) \geq C(P_n) = (n^2 + n)/2$$

$\mathcal{G}$ - an infinite family of graphs

exponential growth

$$\underline{C}(\mathcal{G}) := \liminf_{G \in \mathcal{G}} \sqrt[n]{C(G)} > 1,$$

sub-exponential growth

$$\overline{C}(\mathcal{G}) := \limsup_{G \in \mathcal{G}} \sqrt[n]{C(G)} = 1.$$

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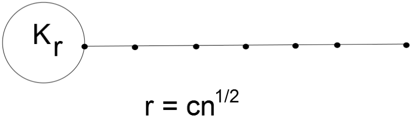
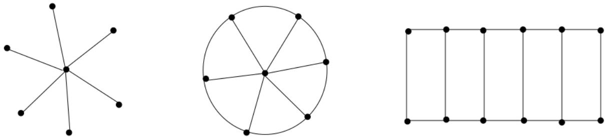
sub-exponential, but not polynomial:






Examples:

Complete graphs, stars, wheels: $C(\mathcal{G}) = 2$

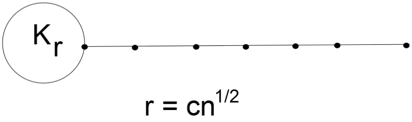
Ladders: $C(\mathcal{G}) = \sqrt{1+\sqrt{2}} \approx 1.55$

Paths and cycles: polynomial growth

All trees have the same density, but

the family of stars is exponential, and

the family of paths is polynomial.




The average degree is a function of $c$ that goes to $\infty$.

## Theorem

*Let $\mathcal{G}$ be an infinite family of connected graphs. If the minimum degree is at least 3 for all $G \in \mathcal{G}$, then $C(\mathcal{G})$ has exponential growth rate.*

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$n_3(G) =$ the number of vertices of degree at least 3 in $G$

**Conjecture 1**

Let $\mathcal{G}$ be an infinite family of connected graphs. If

$$\liminf_{G \in \mathcal{G}} \frac{n_3(G)}{n(G)} > 0,$$

then $\mathcal{G}$ has exponential growth rate.

**Conjecture 1**

Let $\mathcal{G}$ be an infinite family of connected graphs. If

$$\liminf_{G \in \mathcal{G}} \frac{n_3(G)}{n(G)} > 0,$$

then $\mathcal{G}$ has exponential growth rate.

## Theorem

*Conjecture 1 is true for any infinite family of trees and for any infinite family of connected graphs of bounded degree.*

# Connected Partitions of $V$

$P(G) =$ the number of connected partitions of $V$

For any graph $G$ with $n$ vertices,

$$P(G) \leq P(K_n) = B(n) \approx \left(\frac{cn}{\ln n}\right)^n$$

$$P(G) \geq P(\text{tree}) = 2^{n-1}$$

exponential growth

$$\overline{P}(\mathcal{G}) := \limsup_{G \in \mathcal{G}} \sqrt[n]{P(G)} < \infty,$$

super-exponential growth

$$\underline{P}(\mathcal{G}) := \liminf_{G \in \mathcal{G}} \sqrt[n]{P(G)} = \infty.$$

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No families with sub-exponential growth rate:

$$\underline{P}(\mathcal{G}) \geq 2$$

exponential growth

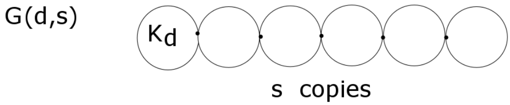
$$\overline{P}(\mathcal{G}) \;:=\; \limsup_{G\in\mathcal{G}} \sqrt[n]{P(G)} \;<\; \infty,$$

super-exponential growth

$$\underline{P}(\mathcal{G}) \;:=\; \liminf_{G\in\mathcal{G}} \;\sqrt[n]{P(G)} \;=\; \infty.$$

| | | |
|---|---|---|
| trees, cycles: | $P(\mathcal{G}) = 2$ | |
| ladders: | $P(\mathcal{G}) = \sqrt{3+\sqrt{10}} \approx 2.48$ | |
| wheels: | $P(\mathcal{G}) = \tau^2 \approx 2.62$ | |
| complete graphs: | super-exponential | $\sqrt[n]{P(K_n)} > \frac{n}{e \ln n}$ |

Another example:




$\mathcal{G}_s = \{G(d,s)) \, : \, d \geq 3\}$ $\mathcal{G}_d = \{G(d,s) \, : \, s \geq 1\}$

$\mathcal{G}_d$ has exponential growth rate $\geq \frac{d}{\ln d}$.

$\mathcal{G}_s$ has super-exponential growth rate.

A family $\mathcal{G}$ is sparse if there is a constant $c$ such that the average degree of $G$ is less than $c$ for all $G \in \mathcal{G}$.

A family $\mathcal{G}$ is dense if, for all positive $c$, the minimum degree of $G$ is greater than $c$ for all but finitely many $G \in \mathcal{G}$.

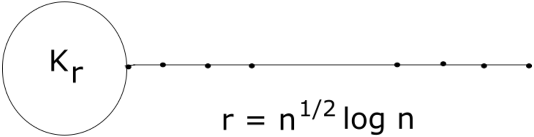
sparse: cycles, wheels, ladders, $\mathcal{G}_d$

dense: complete graphs, $\mathcal{G}_s$

A family $\mathcal{G}$ is sparse if there is a constant $c$ such that the average degree of $G$ is less than $c$ for all $G \in \mathcal{G}$.

### Theorem

*If an infinite family $\mathcal{G}$ is sparse, then $\mathcal{G}$ has exponential growth rate. The converse is false.*



A family $\mathcal{G}$ is dense if, for all positive $c$, the minimum degree of $G$ is greater than $c$ for all but finitely many $G \in \mathcal{G}$.

**Conjecture 3**

If $\mathcal{G}$ is a dense family of graphs, then $\mathcal{G}$ has super-exponential growth rate.

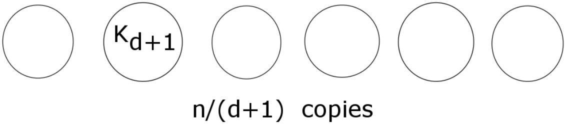
A family $\mathcal{G}$ is dense if, for all positive $c$, the minimum degree of $G$ is greater than $c$ for all but finitely many $G \in \mathcal{G}$.

**Conjecture 3**

If $\mathcal{G}$ is a dense family of graphs, then $\mathcal{G}$ has super-exponential growth rate.

**Conjecture 4**

Given $n$ and $d$, the conjectured graph $G$ minimizing $P(G)$ over all graphs with $n$ vertices and all vertices of degree at least $d$:



# The Average Order of a Connected Set

$\mu_G$ = the average order of a connected set of vertices of a graph $G$

$D_G = \mu(G)/n$

The density is also the probability that a vertex chosen at random from $G$ will belong to a randomly chosen connected set of $G$.

Results for trees:

- (Jameson, 1983)

$$\frac{1}{3} < D_T < 1$$

- For any sequence $\{T_k\}$ such that $D_{T_k} \to 1$, the proportion of vertices of degree 2 tends to 1.

- (V, H. Wang) If all internal vertices of $T$ have degree at least three, then

$$\frac{1}{2} \le D_T < \frac{3}{4}.$$