Triple intersection numbers of metric and cometric association schemes

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Association schemes

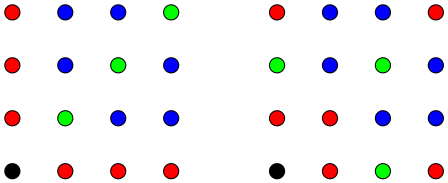
- Let X be a set of vertices and $\mathcal{R} = \{R_0 = \mathrm{id}_X, R_1, \dots, R_d\}$ a set of symmetric relations partitioning X^2.
- $(X, \mathcal{R})$ is said to be a *d-class association scheme* if there exist numbers p_{ij}^h $(0 \le h, i, j \le d)$ such that, for any $x, y \in X$,

$$x \ R_h \ y \Rightarrow |\{z \in X \mid x \ R_i \ z \ R_j \ y\}| = p_{ij}^h$$

- We call the numbers p_{ij}^h $(0 \le h, i, j \le d)$ *intersection numbers*.
- **Problem**: Does an association scheme with given parameters exist? If so, is it unique? Can we determine all such schemes?

Examples

- Hamming schemes: $X = \mathbb{Z}_n^d$, $x \; R_i \; y \Leftrightarrow \text{weight}(x - y) = i$;
- Johnson schemes: $X = \{S \subseteq \mathbb{Z}_n \mid |S| = d\} \quad (2d \leq n)$, $x \; R_i \; y \Leftrightarrow |x \cap y| = d - i$;
- Two schemes with the same parameters:



Bose-Mesner algebra

- Let $A_0, A_1, \ldots A_d$ be binary matrices indexed by X with $(A_i)_{xy} = 1$ *iff* $x \; R_i \; y$.
- These matrices can be diagonalized simultaneously and they share $d+1$ eigenspaces.
- The matrices $\{A_0, A_1, \ldots A_d\}$ are the basis of the *Bose-Mesner algebra* $\mathcal{M}$, which has a second basis $\{E_0, E_1, \ldots E_d\}$ of minimal idempotents for each eigenspace.
- Let P be a $(d+1) \times (d+1)$ matrix with P_{ij} being the eigenvalue of A_j corresponding to the i-th eigenspace.
- Let Q be such that $PQ = |X|I$.
- We call P the *eigenmatrix*, and Q the *dual eigenmatrix*.

Krein parameters

- In the Bose-Mesner algebra $\mathcal{M}$,
  the following relations are satisfied:

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \quad \text{and} \quad E_j = \frac{1}{|X|} \sum_{i=0}^{d} Q_{ij} A_i \ .$$

- We also have

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \quad \text{and} \quad E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{d} q_{ij}^h E_h \ ,$$

  where $\circ$ is the entrywise matrix product.
- The numbers q_{ij}^h are called the *Krein parameters* and are nonnegative algebraic real numbers.

Metric schemes – distance-regular graphs

- If an association scheme satisfies $p_{ij}^h \neq 0 \Rightarrow |i-j| \leq h \leq i+j$ for some ordering of its relations, then it is said to be *metric* or *P-polynomial*.
- Metric association schemes correspond to distance-regular graphs, with $x\ R_i\ y \Leftrightarrow \partial(x,y) = i$.
- The parameters of a metric association scheme can be determined from the *intersection array*

$$\{k, b_1, \dots, b_{d-1}; 1, c_2, \dots, c_d\},$$

where $a_i := p^i_{1,i}$, $b_i := p^i_{1,i+1}$, $c_i := p^i_{1,i-1}$ and $k := b_0 = a_i + b_i + c_i$ $(0 \leq i \leq d)$.

Examples

- Hamming graphs: $b_i = (d-i)(n-1)$, $c_i = i$;
- Johnson graphs: $b_i = (d-i)(n-d-i)$, $c_i = i^2$;
- Pasechnik graphs: $X = (\mathbb{F}_q^3)^2 \times \{+,-\}$, $(x,u,\sigma) \sim (y,v,\tau) \Leftrightarrow \sigma \neq \tau \wedge x-y = u \times v$, intersection array $\{q^3, q^3-1, q^3-q, q^3-q^2+1; 1, q, q^2-1, q^3\}$;
- Coset graphs of Kasami codes: $\{2^{2t+1}-1, 2^{2t+1}-2, 2^{2t}+1; 1, 2, 2^{2t}-1\}$.

Cometric schemes

- If an association scheme satisfies $q_{ij}^h \neq 0 \Rightarrow |i-j| \leq h \leq i+j$ for some ordering of its eigenspaces, then it is said to be *cometric* or *Q-polynomial*.
- The parameters of a cometric association scheme can be determined from the *Krein array*

$$\{m, f_1, \dots, f_{d-1}; 1, g_2, \dots, g_d\},$$

where $e_i := q_{1,i}^i$, $f_i := q_{1,i+1}^i$, $g_i := q_{1,i-1}^i$ and $m := f_0 = e_i + f_i + g_i$ $(0 \leq i \leq d)$.

Examples

- Hamming schemes: $f_i = (d-i)(n-1)$, $g_i = i$;
- Johnson schemes:

$$f_i = \frac{(d-i)n(n-1)(n+1-i)(n-d-i)}{d(n-d)(n+1-2i)(n-2i)}$$

$$g_i = \frac{i(d+1-i)n(n-1)(n-d+1-i)}{d(n-d)(n+2-2i)(n+1-2i)}$$

- Real mutually unbiased bases: $X = \bigcup_{i=1}^{w}(B_i \cup -B_i)$, where $B_1, B_2, \dots, B_w$ are orthornormal bases of $\mathbb{R}^d$ with $\langle x, y \rangle = \pm 1/\sqrt{d}$ for all $x \in B_i$, $y \in B_j$ with $i \neq j$; $x \; R_i \; y \Leftrightarrow \langle x, y \rangle = r_i$, $r_{i=0}^{4} = [1, 1/\sqrt{d}, 0, -1/\sqrt{d}, -1]$; Krein array $\{d, d-1, \frac{d(w-1)}{w}, 1; 1, \frac{d}{w}, d-1, d\}$;
- Codewords of dual Kasami codes: $\{2^{2t+1}-1, 2^{2t+1}-2, 2^{2t}+1; 1, 2, 2^{2t}-1\}$.

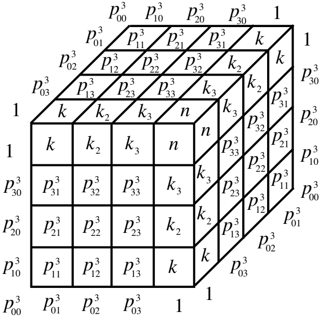
Triple intersection numbers

- In an association scheme,
  the intersection numbers p^h_{ij} only depend on h, i, j.
- Let $x, y, z \in X$ with $x \; R_W \; y$, $x \; R_V \; z$ and $y \; R_U \; z$.
- We define *triple intersection numbers* as
  $\begin{bmatrix} x & y & z \\ h & i & j \end{bmatrix} := |\{w \in X \mid w \; R_h \; x, w \; R_i \; y, w \; R_j \; z\}|.$
- $\begin{bmatrix} x & y & z \\ h & i & j \end{bmatrix}$ may depend on the particular choice of x, y, z!
- When x, y, z are fixed, we abbreviate $\begin{bmatrix} x & y & z \\ h & i & j \end{bmatrix}$ as $[h \; i \; j]$.

Computing triple intersection numbers

- We have $3d^2$ equations connecting triple intersection numbers to p^h_{ij}:

$$\sum_{\ell=1}^{d} [\ell \ i \ j] = p^U_{ij} - \delta_{iW}\delta_{jV},$$
$$\sum_{\ell=1}^{d} [h \ \ell \ j] = p^V_{hj} - \delta_{hW}\delta_{jU},$$
$$\sum_{\ell=1}^{d} [h \ i \ \ell] = p^W_{hi} - \delta_{hV}\delta_{iU}.$$



- All triple intersection numbers are nonnegative integers.

Krein condition

- **Theorem** ([BCN89, Theorem 2.3.2], [CJ08, Theorem 3]):
  Let $(X, \mathcal{R})$ be a d-class association scheme,
  Q its dual eigenmatrix, and q_{ij}^h its Krein parameters.

- $q_{ij}^h = 0$ iff for all triples $x, y, z \in X$:

$$\sum_{r,s,t=0}^{d} Q_{rh} Q_{si} Q_{tj} \begin{bmatrix} x & y & z \\ r & s & t \end{bmatrix} = 0$$

- This gives a new equation
  in terms of triple intersection numbers.

- The `sage-drg` [Vid18] Sage package can use all of the above
  to determine the possible triple intersection numbers.

Tables of feasible parameters

- Various lists of feasible intersection arrays for distance-regular graphs have been published [BCN89, BCN94, Bro11].
- Recently, Williford [Wil17] has published lists of feasible Krein arrays of Q-polynomial association schemes:
  - 3-class primitive Q-polynomial association schemes on up to 2800 vertices: 62 known examples, 359 open cases;
  - 4-class Q-bipartite, but not Q-antipodal, Q-polynomial association schemes on up to 10000 vertices: 19 known examples, 488 open cases; and
  - 5-class Q-bipartite, but not Q-antipodal, Q-polynomial association schemes on up to 50000 vertices: 7 known examples, 16 open cases.

Nonexistence results

- We have used integer linear programming to find possible triple intersection numbers for the open cases in the lists.
- We have been able to prove nonexistence for
  - 31 cases of 3-class Q-polynomial association schemes, of which 8 correspond to distance-regular graphs,
  - 92 cases of 4-class Q-polynomial association schemes,
  - 12 cases of 5-class Q-polynomial association schemes, of which one corresponds to a distance-regular graph, and
  - one case of a diameter 3 non-Q-polynomial distance-regular graph.

Closed cases of Q-polynomial association schemes

- Smallest closed case:
  $\{12, 338/35, 39/25; 1, 312/175, 39/5\}$ on 91 vertices.
- Double counting has been used to settle the cases
  - $\{24, 20, 36/11; 1, 30/11, 24\}$ on 225 vertices,
  - $\{104, 70, 25; 1, 7, 80\}$ on 1470 vertices, and
  - $\{132, 343/3, 56, 28/3, 1; 1, 28/3, 56, 343/3, 132\}$ on 3500 vertices.
- Most remaining cases have been ruled out because there was no integral nonnegative solution for triple intersection numbers corresponding to a triple of vertices at some given relations.
- Next remaining open case:
  $\{14, 108/11, 15/4; 1, 24/11, 45/4\}$ on 99 vertices.

Infinite families

We have been able to extend the nonexistence results to the following infinite families of Q-polynomial association schemes:

- distance-regular graphs with intersection arrays

$$\{(2r+1)(4r+1)(4t-1), 8r(4rt-r+2t), (r+t)(4r+1); 1, (r+t)(4r+1), 4r(2r+1)(4t-1)\}$$

- association schemes with Krein arrays $\{2r^2-1, 2r^2-2, r^2+1; 1, 2, r^2-1\}$ (r odd),
- association schemes with Krein arrays $\{r^3, r^3-1, r^3-r, r^3-r^2+1; 1, r, r^2-1, r^3\}$ (r odd), and
- association schemes with Krein arrays $\left\{\frac{r^2+1}{2}, \frac{r^2-1}{2}, \frac{(r^2+1)^2}{2r(r+1)}, \frac{(r-1)(r^2+1)}{4r}, \frac{r^2+1}{2r}; 1, \frac{(r-1)(r^2+1)}{2r(r+1)}, \frac{(r+1)(r^2+1)}{4r}, \frac{(r-1)(r^2+1)}{2r}, \frac{r^2+1}{2}\right\}$ ($r \equiv 3 \pmod 4$).

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