Tridiagonal pairs of Racah type, the double lowering operator Ψ, and the universal enveloping algebra $U(\mathfrak{sl}_2)$

TerwilligerFest - Combinatorics around the q-Onsager Algebra

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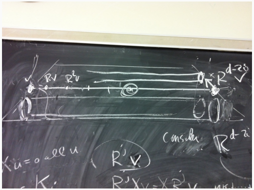
★Happy birthday, Paul! ★



★Thank you for the many years ★ of mathematical adventures!

Introduction & Motivation

This talk is about **tridiagonal pairs** and **tridiagonal systems**.

Broadly speaking, I am interested in how the 8 tridiagonal systems associated with a tridiagonal pair are related to one another.

To study this, we can associate some interesting and meaningful linear transformations with each tridiagonal system.

Understanding how these different maps relate to one another is one way to better understand what's going on the tridiagonal systems.

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- q-Racah case

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- Racah case $\longleftarrow$???

Definition of a tridiagonal pair

Let $\mathbb{K}$ denote a field.

Let V denote a vector space over $\mathbb{K}$ of finite positive dimension.

By a **tridiagonal pair** (or TD pair) on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ satisfying:

1. Each of A, A^* is diagonalizable.
2. There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that
$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$
where $V_{-1} = 0$ and $V_{d+1} = 0$.
3. There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that
$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta),$$
where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.
4. There does not exist a subspace W of V such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

Example: Q-polynomial distance-regular graph

Let $\Gamma = \Gamma(X, E)$ denote a Q-polynomial distance-regular graph.

Let A denote the adjacency matrix of Γ.

Fix $x \in X$. Let $A^* = A^*(x)$ denote the dual adjacency matrix of Γ with respect to x. That is, A^* is the diagonal matrix in $Mat_{\mathbb{C}}(X)$ whose (y, y)-entry is given by

$$(A^*)_{yy} = |X| \, (E_1)_{xy} \, .$$

Let W denote an irreducible (A, A^*)-submodule of $\mathbb{C}^{|X|}$.

Then A, A^* form a TD pair on W.

Standard ordering

Definition
Given a TD pair A, A^*, an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A is called **standard** whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$. (A similar discussion applies to A^*.)

Lemma (Ito, Terwilliger)
If $\{V_i\}_{i=0}^d$ is a standard ordering of the eigenspaces of A, then $\{V_{d-i}\}_{i=0}^d$ is standard and no other ordering is standard.

Tridiagonal system

By a **tridiagonal system** (or TD system) on V, we mean a sequence

$$\Phi = (A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$$

that satisfies (1)–(3) below.

1. A, A^* is a tridiagonal pair on V.
2. $\{V_i\}_{i=0}^d$ is a standard ordering of the eigenspaces of A.
3. $\{V_i^*\}_{i=0}^d$ is a standard ordering of the eigenspaces of A^*.

Relatives of a TD system

A given TD system can be modified in a number of ways to get a new TD system.

$$(A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d) \qquad (A^*; \{V_i^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d)$$
$$(A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d) \qquad (A^*; \{V_{d-i}^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d)$$
$$(A; \{V_i\}_{i=0}^d; A^*; \{V_{d-i}^*\}_{i=0}^d) \qquad (A^*; \{V_i^*\}_{i=0}^d; A; \{V_{d-i}\}_{i=0}^d)$$
$$(A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_{d-i}^*\}_{i=0}^d) \qquad (A^*; \{V_{d-i}^*\}_{i=0}^d; A; \{V_{d-i}\}_{i=0}^d)$$

These eight TD systems are said to be **relatives** of one another.

Assumptions

Until further notice, we fix a TD system

$$\Phi = (A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d).$$

Let

$$\Phi^{\Downarrow} = (A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d).$$

Throughout this talk, we will focus on Φ and its associated objects. Keep in mind that a similar discussion applies to $\Phi^{\Downarrow}$ and its associated objects.

For any object f associated with Φ, we let $f^{\Downarrow}$ denote the corresponding object associated with $\Phi^{\Downarrow}$.

The Racah case

For $0 \le i \le d$, we let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) corresponding to the eigenspace V_i (resp. V^*_i).

Definition

We say that the TD system Φ has **Racah type** whenever there exist scalars $a, b, c, a^*, b^*, c^* \in \mathbb{K}$ such that c, c^* are nonzero and

$$\theta_i = a + bi + ci^2,$$
$$\theta^*_i = a^* + b^* i + c^* i^2$$

for $0 \le i \le d$.

Assumption:

Throughout this talk, we assume that Φ has Racah type.

The split decompositions of V

Definition

For $0 \le i \le d$, define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d),$$
$$U_i^{\Downarrow} = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_{d-i}).$$

Theorem (Ito, Tanabe, Terwilliger)

$$V = U_0 + U_1 + \cdots + U_d \qquad \textit{(direct sum)}$$
$$V = U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_d^{\Downarrow} \qquad \textit{(direct sum)}$$

We refer to $\{U_i\}_{i=0}^d$ *as the* **first split decomposition** *of* V.

We refer to $\{U_i^{\Downarrow}\}_{i=0}^d$ *as the* **second split decomposition** *of* V.

The maps $H, H^{\Downarrow}$

Definition

Let $H : V \to V$ denote the linear transformation such that for $0 \leq i \leq d$, U_i is an eigenspace of H with eigenvalue $d - 2i$. That is,

$$(H - (d - 2i)I)U_i = 0$$

for $0 \leq i \leq d$.

Definition

Let $H^{\Downarrow} : V \to V$ denote the linear transformation such that for $0 \leq i \leq d$, $U_i^{\Downarrow}$ is an eigenspace of $H^{\Downarrow}$ with eigenvalue $d - 2i$. That is,

$$(H^{\Downarrow} - (d - 2i)I)U_i^{\Downarrow} = 0$$

for $0 \leq i \leq d$.

Split decompositions of V

Lemma (Ito, Tanabe, Terwilliger)

Let $0 \leq i \leq d$.

A, A^* *act on the first split decomposition in the following way:*

$$(A - \theta_i I)U_i \subseteq U_{i+1}, \qquad (A^* - \theta^*_i I)U_i \subseteq U_{i-1}.$$

A, A^* *act on the second split decomposition in the following way:*

$$(A - \theta_{d-i} I)U^{\Downarrow}_i \subseteq U^{\Downarrow}_{i+1}, \qquad (A^* - \theta^*_i I)U^{\Downarrow}_i \subseteq U^{\Downarrow}_{i-1}.$$

The raising maps $R, R^{\Downarrow}$

Let

$$R = A - aI + \frac{b}{2}(H - dI) - \frac{c}{4}(H - dI)^2 .$$

By construction,

- R acts on U_i as $A - \theta_i I$ for $0 \le i \le d$,
- $RU_i \subseteq U_{i+1} \qquad (0 \le i < d), \quad RU_d = 0.$

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Let

$$R^{\Downarrow} = A - aI - \frac{b}{2}\left(H^{\Downarrow} + dI\right) - \frac{c}{4}\left(H^{\Downarrow} + dI\right)^2.$$

By construction,

- $R^{\Downarrow}$ acts on $U_i^{\Downarrow}$ as $A - \theta_{d-i} I$ for $0 \le i \le d$,
- $R^{\Downarrow} U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow} \qquad (0 \le i < d), \quad R^{\Downarrow} U_d^{\Downarrow} = 0.$

Relating R and H, $R^{\Downarrow}$ and $H^{\Downarrow}$

Lemma

Both

$$HR - RH = -2R,$$
$$H^{\Downarrow}R^{\Downarrow} - R^{\Downarrow}H^{\Downarrow} = -2R^{\Downarrow}$$

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$$H^{\Downarrow}R^{\Downarrow} - R^{\Downarrow}H^{\Downarrow} = -2R^{\Downarrow}$$

Corollary

Both

$$\frac{AH - HA}{2} = A - aI + \frac{b}{2}(H - dI) - \frac{c}{4}(H - dI)^2,$$

$$\frac{AH^{\Downarrow} - H^{\Downarrow}A}{2} = A - aI - \frac{b}{2}(H^{\Downarrow} + dI) - \frac{c}{4}(H^{\Downarrow} + dI)^2.$$

The linear transformation ψ

We now introduce the linear transformation $\psi : V \to V$.
The exact definition is a bit technical. One key feature of ψ is the following.

Lemma (B. 2012)

For $0 \leq i \leq d$, *both*

$$\psi U_i \subseteq U_{i-1},$$
$$\psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow}.$$

Moreover, $\psi^{d+1} = 0$.

In light of the above result, we refer to ψ as the **double lowering operator.**

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We have that $\psi = \psi^{\Downarrow}$.

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$$H^{\Downarrow}\psi - \psi H^{\Downarrow} = 2\psi.$$

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Lemma

Both

$$\psi R - R\psi = H,$$
$$\psi R^{\Downarrow} - R^{\Downarrow}\psi = H^{\Downarrow}.$$

Relation summary

Our relations

$$H\psi - \psi H = 2\psi$$
$$HR - RH = -2R$$
$$\psi R - R\psi = H$$

$\Phi^{\Downarrow}$-analogues

$$H^{\Downarrow}\psi - \psi H^{\Downarrow} = 2\psi$$
$$H^{\Downarrow}R^{\Downarrow} - R^{\Downarrow}H^{\Downarrow} = -2R^{\Downarrow}$$
$$\psi R^{\Downarrow} - R^{\Downarrow}\psi = H^{\Downarrow}$$

The universal enveloping algebra $U(\mathfrak{sl}_2)$

Definition

The **universal enveloping algebra** $U(\mathfrak{sl}_2)$ is defined to be the unital associative $\mathbb{K}$-algebra with generators

$$e, \qquad f, \qquad h$$

and relations

$$\begin{aligned} ef - fe &= h, \\ he - eh &= 2e, \\ hf - fh &= -2f. \end{aligned}$$

Two $U(\mathfrak{sl}_2)$-module structures on V

Theorem

There exists a $U(\mathfrak{sl}_2)$-module structure on V for which the generators act as follows:

element of $U(\mathfrak{sl}_2)$	e	f	h
action on V	ψ	R	H

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action on V	ψ	$R^{\Downarrow}$	$H^{\Downarrow}$

Irreducible $U(\mathfrak{sl}_2)$-submodules of V

Let denote M the subalgebra of $\mathrm{End}(V)$ generated by A.

For $0 \le i \le d/2$ and $v \in K_i = U_i \cap U_i^{\Downarrow}$, let Mv denote the M-submodule of V generated by v.

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Lemma

Let $0 \le i \le d/2$ and $v \in K_i$. For either of the $U(\mathfrak{sl}_2)$-actions on V from the previous slide, Mv is an irreducible $U(\mathfrak{sl}_2)$-submodule of V with dimension $d - 2i + 1$.

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Lemma

For either of the $U(\mathfrak{sl}_2)$-actions on V from the previous slide, V can be written as a direct sum of its irreducible $U(\mathfrak{sl}_2)$-submodules.

The Casimir element of $U(\mathfrak{sl}_2)$

Definition

Define the normalized Casimir element C of $U(\mathfrak{sl}_2)$ by

$$\begin{aligned} C &= (h+1)^2 + 4fe, \\ &= (h-1)^2 + 4ef. \end{aligned}$$

We mention that

- C is in the center of $U(\mathfrak{sl}_2)$.
- C acts as a scalar multiple of the identity on irreducible $U(\mathfrak{sl}_2)$-modules.

The action of the Casimir element on V

Lemma

With respect to the first $U(\mathfrak{sl}_2)$-module structure on V, the action of the C on V is equal to both

$$(H+1)^2 + 4R\psi,$$
$$(H-1)^2 + 4\psi R.$$

Lemma

With respect to the second $U(\mathfrak{sl}_2)$-module structure on V, the action of the C on V is equal to both

$$(H^{\Downarrow}+1)^2 + 4R^{\Downarrow}\psi,$$
$$(H^{\Downarrow}-1)^2 + 4\psi R^{\Downarrow}.$$

The action of the Casimir element on V

Lemma

Let $0 \leq i \leq d/2$ and $v \in K_i$.

With respect to either of the $U(\mathfrak{sl}_2)$-module structures on V, C acts on Mv as $d - 2i + 1$ times the identity.

Comparing the actions of C on V

Lemma

The following coincide:

(i) *the action of C on V for the first $U(\mathfrak{sl}_2)$-module structure on V,*

(ii) *the action of C on V for the second $U(\mathfrak{sl}_2)$-module structure on V.*

Comparing the actions of C on V

Lemma

The following coincide:

(i) *the action of C on V for the first $U(\mathfrak{sl}_2)$-module structure on V,*

(ii) *the action of C on V for the second $U(\mathfrak{sl}_2)$-module structure on V.*

Corollary

The following four expressions are equal:

$$(H+1)^2 + 4R\psi,$$
$$(H-1)^2 + 4\psi R,$$
$$(H^{\Downarrow}+1)^2 + 4R^{\Downarrow}\psi,$$
$$(H^{\Downarrow}-1)^2 + 4\psi R^{\Downarrow}.$$

The relationship between $\psi, H, H^{\Downarrow}$

Proposition

The following coincide:

$$(I - c\psi)H + (b + cd)\psi, \qquad (I - c\psi)H^{\Downarrow} - (b + cd)\psi,$$
$$H(I - c\psi) + (b + cd + 2)\psi, \qquad H^{\Downarrow}(I - c\psi) - (b + cd - 2)\psi.$$

The relationship between $\psi, H, H^{\Downarrow}$

Lemma

The element $I - c\psi \in \mathrm{End}(V)$ *is invertible. Its inverse is as follows:*

$$(I - c\psi)^{-1} = \sum_{i=0}^{d} c^i \psi^i.$$

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Theorem

We have

$$H - H^{\Downarrow} = \frac{-2(b + cd)\psi}{I - c\psi}.$$

The denominator above is invertible by the previous lemma.

Lemma

The operators ψ and $H - H^{\Downarrow}$ commute.

The relationship between $\psi, H, H^{\Downarrow}$

Lemma

For $0 \leq i \leq d$,

$$\left(H - H^{\Downarrow}\right) U_i \subseteq U_0 + U_1 + \cdots + U_{i-1},$$
$$\left(H - H^{\Downarrow}\right) U_i^{\Downarrow} \subseteq U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}.$$

Moreover $H - H^{\Downarrow}$ is nilpotent.

Lemma

The element $c\left(H - H^{\Downarrow}\right) - 2(b + cd)I \in \mathrm{End}(V)$ is invertible.

The relationship between $\psi, H, H^{\Downarrow}$

Theorem

We have

$$\psi = \frac{H - H^{\Downarrow}}{c\,(H - H^{\Downarrow}) - 2(b + cd)I}.$$

The denominator above is invertible by the previous lemma.

Future Work

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Future Work

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- The equations we found are analogous to those from the q-Racah case, so it seems reasonable to hope that Δ can be written as a product of two polynomials in ψ. Now we just need to find it!
- Wish Paul happy birthday one more time.

Thank you for your attention!