# A uniform approach to the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$ and generalized reflection equations


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# Overview

This talk is about the **positive part** of the $q$-deformed enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$. This positive part is denoted by $U_q^+$.

The algebra $U_q^+$ admits an embedding into a $q$-shuffle algebra. This embedding is due to **Rosso**.

Using the Rosso embedding, we will discuss a **uniform approach** to three PBW bases for $U_q^+$ due to **Damiani**, **Beck**, and **Terwillger**.

We will also present a **generalized reflection equation** of the form $RK\widehat{R}K$, where the $K$-operators are given in closed matrix form using the uniform approach.

# 1 The algebra $U_q^+$ and the Rosso embedding

# Preliminaries

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Let $\mathbb{F}$ denote an algebraically closed field of characteristic zero.

All algebras in this paper are associative, over $\mathbb{F}$, and have a multiplicative identity.

Let $q$ denote a nonzero scalar in $\mathbb{F}$ that is not a root of unity.

For $n \in \mathbb{Z}$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

# The algebra $U_q^+$

**Definition**

Let $U_q^+$ denote the algebra with generators $A, B$ and the $q$-**Serre relations**

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = 0,$$

$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = 0.$$

The algebra $U_q^+$ is called the **positive part** of $U_q(\widehat{\mathfrak{sl}}_2)$.

The algebra $U_q^+$ is infinite-dimensional and noncommutative.

# The $q$-shuffle algebra $\mathbb{V}$

We now recall an embedding of $U_q^+$ in to a $q$-**shuffle algebra** $\mathbb{V}$. This embedding was introduced by **Rosso** in 1995.

The underlying vector space for $\mathbb{V}$ is the free algebra with two noncommuting generators $x, y$.

We call $x, y$ **letters**. A product $v_1 v_2 \cdots v_n$ of letters is called a **word**, and $n$ is called the **length** of this word.

The word of length 0 is called **trivial** and denoted by $\mathbb{1}$.

The collection of all words form a basis for the vector space $\mathbb{V}$, called the **standard basis**.

# The $q$-shuffle product on $\mathbb{V}$

We now define the $q$-shuffle product $\star$ on $\mathbb{V}$. To describe it, we use the bilinear form

$$\langle x, x\rangle = \langle y, y\rangle = 2, \qquad \langle x, y\rangle = \langle y, x\rangle = -2.$$

We first display the $q$-shuffle product of two letters $a, b$. We have

$$a \star b = ab + baq^{\langle a,b\rangle}.$$

In other words,

$$x \star x = xx + q^2 xx, \qquad x \star y = xy + q^{-2} yx,$$

$$y \star x = yx + q^{-2} xy, \qquad y \star y = yy + q^2 yy.$$

# The $q$-shuffle product on $\mathbb{V}$, cont.

We now describe the $q$-shuffle product of two words. Consider an example $a_1 a_2 \star b_1 b_2$.

This product is a linear combination of words. Each word is obtained by shuffling the $a_i$'s into the $b_j$'s, and its coefficient is determined by how this word is obtained: there's a cost of $q^{\langle a_i, b_j \rangle}$ each time an $a_i$ moves past a $b_j$. Therefore,

$$\begin{aligned} a_1 a_2 \star b_1 b_2 &= a_1 a_2 b_1 b_2 \\ &+ a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle} \\ &+ a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle} \\ &+ b_1 a_1 a_2 b_2 q^{\langle a_2, b_1 \rangle + \langle a_1, b_1 \rangle} \\ &+ b_1 a_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle + \langle a_1, b_1 \rangle} \\ &+ b_1 b_2 a_1 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle + \langle a_1, b_1 \rangle + \langle a_1, b_2 \rangle}. \end{aligned}$$

# The $q$-shuffle algebra $\mathbb{V}$

**Theorem (Rosso 1995)**

*The vector space $\mathbb{V}$, equipped with the $q$-shuffle product $\star$, becomes an algebra.*

We emphasize that the $q$-shuffle algebra is associative and noncommutative.

# The embedding of $U_q^+$ into $\mathbb{V}$

## Definition

Let $U$ denote the subalgebra of the $q$-shuffle algebra $\mathbb{V}$ generated by $x, y$.

## Theorem (Rosso 1995)

*There exists an algebra isomorphsim* $\natural : U_q^+ \to U$ *that sends* $A \mapsto x$ *and* $B \mapsto y$.

Throughout this talk, we identify $U_q^+$ with $U$ via $\natural$.

# 2 A uniform approach

# Three PBW bases for $U_q^+$

In the literature, there are three **Poincaré-Birkhoff-Witt basis** (or **PBW basis**) for $U_q^+$, due to **Damiani**, **Beck**, and **Terwilliger** respectively.

Closed form for the Damiani and the Beck PBW basis elements has been found by Terwilliger using the Rosso embedding.

In addition, Terwilliger obtained the alternating PBW basis where the basis elements are given in closed form using the Rosso embedding.

In order to display these PBW bases, we need the notion of a Catalan word.

# Catalan words

**Definition**

Define $\overline{x} = 1$ and $\overline{y} = -1$. A word $a_1 \cdots a_k$ is **Catalan** if

$$\overline{a}_1 + \cdots + \overline{a}_i \geq 0 \quad (1 \leq i \leq k-1),$$

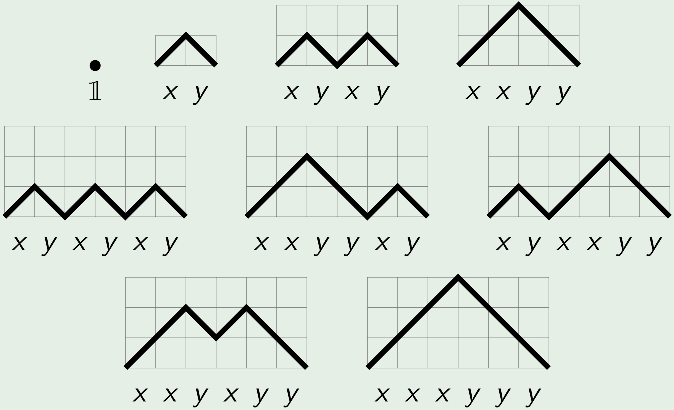
$$\overline{a}_1 + \cdots + \overline{a}_k = 0.$$

In this case $k$ is even.

# Catalan words, cont.

## Example

We give the Catalan words of length $\leq 6$.



# The Damiani PBW basis under $\natural$

## Definition (Terwilliger 2018)

For $n \in \mathbb{N}$, define

$$C_n = \sum a_1 \cdots a_{2n} \prod_{i=1}^{2n} [1 + \overline{a}_1 + \cdots + \overline{a}_i]_q,$$

where the sum is over all Catalan words $a_1 \cdots a_{2n}$ of length $2n$.

## Example

We list $C_n$ for $0 \leq n \leq 3$.

$$C_0 = \mathbb{1}, \qquad C_1 = [2]_q xy, \qquad C_2 = [2]_q^2 xyxy + [2]_q^2 [3]_q xxyy,$$

$$C_3 = [2]_q^3 xyxyxy + [2]_q^3 [3]_q xxyyxy + [2]_q^3 [3]_q xyxxyy + [2]_q^3 [3]_q^2 xxyxyy + [2]_q^2 [3]_q^2 [4]_q xxxyyy.$$

# The Damiani PBW basis under $\natural$, cont.

**Theorem (Terwilliger 2018)**

*The elements* $\{xC_n\}_{n\in\mathbb{N}}, \{C_n y\}_{n\in\mathbb{N}}, \{C_n\}_{n\in\mathbb{N}^+}$ *form a PBW basis for* $U$.

This is the Damiani PBW basis under $\natural$, up to some scalars.

# The Beck PBW basis under $\natural$

**Theorem (Terwilliger 2021)**

*The elements* $\{xC_n\}_{n\in\mathbb{N}}, \{C_n y\}_{n\in\mathbb{N}}, \{xC_n y\}_{n\in\mathbb{N}}$ *form a PBW basis for* $U$.

This is the Beck PBW basis under $\natural$, up to some scalars.

# The alternating PBW basis for $U$

**Definition (Terwilliger 2018)**

We define

$$W_0 = x, \quad W_{-1} = xyx, \quad W_{-2} = xyxyx, \quad W_{-3} = xyxyxyx, \quad \cdots,$$

$$W_1 = y, \quad W_2 = yxy, \quad W_3 = yxyxy, \quad W_4 = yxyxyxy, \quad \cdots,$$

$$\tilde{G}_1 = xy, \quad \tilde{G}_2 = xyxy, \quad \tilde{G}_3 = xyxyxy, \quad \tilde{G}_4 = xyxyxyxy, \quad \cdots,$$

$$G_1 = yx, \quad G_2 = yxyx, \quad G_3 = yxyxyx, \quad G_4 = yxyxyxyx, \quad \cdots.$$

The above words are called **alternating**.

# The alternating PBW basis for $U$, cont.

**Theorem (Terwilliger 2018)**

*Each of the following form a PBW basis for $U$ under an appropriate linear ordering:*

- $\{W_{-n}\}_{n\in\mathbb{N}}, \qquad \{W_{n+1}\}_{n\in\mathbb{N}}, \qquad \{\tilde{G}_n\}_{n=1}^{\infty};$
- $\{W_{-n}\}_{n\in\mathbb{N}}, \qquad \{W_{n+1}\}_{n\in\mathbb{N}}, \qquad \{G_n\}_{n=1}^{\infty}.$

# The generating functions

Let $\tilde{G}_0 = \mathbb{1}$. We will review how the elements $\{C_n\}_{n\in\mathbb{N}}$ and $\{\tilde{G}_n\}_{n\in\mathbb{N}}$ are related.

Define the generating functions

$$C(t) = \sum_{n\in\mathbb{N}} C_n t^n, \qquad \tilde{G}(t) = \sum_{n\in\mathbb{N}} \tilde{G}_n t^n.$$

**Lemma (Terwilliger 2018)**

*We have*

$$\tilde{G}(-qt) \star C(t) \star \tilde{G}(-q^{-1}t) = \mathbb{1}\,.$$

# The generating functions, cont.

Let $D(t)$ denote the multiplicative inverse of $\tilde{G}(t)$. Write

$$D(t) = \sum_{n \in \mathbb{N}} D_n t^n.$$

**Theorem (Ruan 2023)**

*For $n \in \mathbb{N}$,*

$$D_n = (-1)^n \sum a_1 \cdots a_{2n} \prod_{i=1}^{2n} [\overline{a}_1 + \cdots + \overline{a}_{i-1} + (\overline{a}_i + 1)/2]_q,$$

*where the sum is over all Catalan words $a_1 \cdots a_{2n}$ of length $2n$.*

# Patterns in $C_n$, $D_n$, $\tilde{G}_n$

We compare the elements $C_n$ and $D_n$ for $n \in \mathbb{N}$. Recall that

$$C_n = \sum a_1 \cdots a_{2n} \prod_{i=1}^{2n} [\overline{a}_1 + \cdots + \overline{a}_{i-1} + \overline{a}_i + 1]_q,$$

$$D_n = (-1)^n \sum a_1 \cdots a_{2n} \prod_{i=1}^{2n} [\overline{a}_1 + \cdots + \overline{a}_{i-1} + (\overline{a}_i + 1)/2]_q,$$

where the sums are over all Catalan words $a_1 \cdots a_{2n}$ of length $2n$.

### Definition

For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, define

$$\Delta_n^{(m)} = \sum a_1 \cdots a_{2n} \prod_{i=1}^{2n} [\overline{a}_1 + \cdots + \overline{a}_{i-1} + m(\overline{a}_i + 1)/2]_q,$$

where the sum is over all Catalan words $a_1 \cdots a_{2n}$ of length $2n$.

# Patterns in $C_n$, $D_n$, $\tilde{G}_n$, cont.

## Example

For $n \in \mathbb{N}$,

$$\Delta_n^{(2)} = C_n, \qquad \Delta_n^{(1)} = (-1)^n D_n, \qquad \Delta_n^{(-1)} = (-1)^n \tilde{G}_n,$$

$$\Delta_n^{(0)} = \begin{cases} \mathbb{1}, & \text{if } n = 0; \\ 0, & \text{if } n \geq 1. \end{cases}$$

# A exponential formula

For notational convenience, for $m \in \mathbb{Z}$ we write

$$\Delta^{(m)}(t) = \sum_{n \in \mathbb{N}} \Delta_n^{(m)} t^n.$$

**Example**

We have

$$\Delta^{(2)}(t) = C(t), \qquad \Delta^{(1)}(t) = D(-t), \qquad \Delta^{(-1)}(t) = \tilde{G}(-t),$$

$$\Delta^{(0)}(t) = \mathbb{1}\,.$$

# A exponential formula, cont.

**Theorem (Ruan 2025)**

*For $m \in \mathbb{Z}$,*

$$\Delta^{(m)}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{[mn]_q}{n} x C_{n-1} y t^n\right).$$

# Some special cases

In the previous theorem, if we set $m = 2, -1, 1$ then we get the following known formulas obtained by **Beck**, **Chari**, **Pressley**; **Terwilliger**.

$$C(t) = \exp\left(\sum_{n=1}^{\infty} \frac{[2n]_q}{n} x C_{n-1} y t^n\right),$$

$$\tilde{G}(-t) = \exp\left(-\sum_{n=1}^{\infty} \frac{[n]_q}{n} x C_{n-1} y t^n\right),$$

$$D(-t) = \exp\left(\sum_{n=1}^{\infty} \frac{[n]_q}{n} x C_{n-1} y t^n\right).$$

# A corollary

**Corollary (Ruan 2025)**

*For $m \in \mathbb{Z}$,*

$$\Delta^{(m)}(t) = \begin{cases} D(-q^{1-m}t) \star D(-q^{3-m}t) \star \cdots \star D(-q^{m-1}t), & \text{if } m \geq 1; \\ \mathbb{1}, & \text{if } m = 0; \\ \tilde{G}(-q^{1+m}t) \star \tilde{G}(-q^{3+m}t) \star \cdots \star \tilde{G}(-q^{-m-1}t), & \text{if } m \leq -1. \end{cases}$$

# 3 Generalized reflection equations

# Motivation from the $q$-Onsager algebra

We now discuss an application of the uniform approach.

In 2023, **Lemarthe**, **Baseilhac**, and **Gainutdinov** obtained a reflection equation over the $q$-Onsager algebra of the form $RKRK$.

The $R$-matrix $R^{(j_1,j_2)}(t)$ ($j_1, j_2 \in \frac{1}{2}\mathbb{N}^+$) is constructed from $R^{(\frac{1}{2},\frac{1}{2})}$ using a fused recurrence relation.

The $K$-matrix $K^{(j)}(t)$ ($j \in \frac{1}{2}\mathbb{N}^+$) is constructed from $K^{(\frac{1}{2})}$ using a fused recurrence relation.

# Motivation from the $q$-Onsager algebra, cont.

Motivated by the result from the previous slide, we obtained a generalized reflection equation over $U_q^+$ of the form $RK\widehat{R}K$.

Here the $R$-matrix is the same as before, the $\widehat{R}$-matrix is a diagonal matrix, and the $K$-matrix is given in closed matrix form using the uniform approach.

We will present this generalized reflection equation for the rest of this talk.

# A few remarks on the $R$-matrix

The matrices $R^{(j_1,j_2)}(t)$ ($j_1, j_2 \in \frac{1}{2}\mathbb{N}^+$) have been studied extensively in the theory of integrable systems.

It is known that they satisfy the **Yang-Baxter equation**

$$R_{12}^{(j_1,j_2)}(t_1/t_2)R_{13}^{(j_1,j_3)}(t_1/t_3)R_{23}^{(j_2,j_3)}(t_2/t_3)$$
$$=R_{23}^{(j_2,j_3)}(t_2/t_3)R_{13}^{(j_1,j_3)}(t_1/t_3)R_{12}^{(j_1,j_2)}(t_1/t_2).$$

## Example

We have

$$R^{(\frac{1}{2},\frac{1}{2})}(t) = \begin{pmatrix} qt - q^{-1}t^{-1} & 0 & 0 & 0 \\ 0 & t - t^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & t - t^{-1} & 0 \\ 0 & 0 & 0 & qt - q^{-1}t^{-1} \end{pmatrix}.$$

# Defining the $\widehat{R}$-matrix

**Definition**

For $j \in \frac{1}{2}\mathbb{N}^+$, define $\omega^{(j)} = \mathrm{diag}(q^j, q^{j-1}, \ldots, q^{-j})$.

For $j_1, j_2 \in \frac{1}{2}\mathbb{N}^+$, define

$$\widehat{R}^{(j_1,j_2)} = q^{-2j_1j_2}\,\mathrm{diag}\left((\omega^{(j_2)})^{2j_1}, (\omega^{(j_2)})^{2j_1-2}, \ldots, (\omega^{(j_2)})^{-2j_1}\right).$$

Both $R^{(j_1,j_2)}$ and $\widehat{R}^{(j_1,j_2)}$ have size $(2j_1+1)(2j_2+1) \times (2j_1+1)(2j_2+1)$.

**Example**

We have

$$\widehat{R}^{(\frac{1}{2},j_2)} = \mathrm{diag}\left(1, q^{-1}, \ldots, q^{-2j_2}, q^{-2j_2}, \ldots, q^{-1}, 1\right).$$

# Two operators on $\mathbb{V}$

In order to define the $K$-matrix, we introduce the following operators $x^{-1}, y^{-1}$ on $\mathbb{V}$.

**Definition**

For $n \geq 1$ and a word $w = a_1 \cdots a_n$, define

$$wy^{-1} = \begin{cases} 0, & \text{if } a_n = x; \\ a_1 \cdots a_{n-1}, & \text{if } a_n = y. \end{cases}$$

By convention, $\mathbb{1}\, y^{-1} = 0$.

For $v \in \mathbb{V}$, we define $vy^{-1}$ linearly.

**Definition**

For $v \in \mathbb{V}$, we define $x^{-1}v$ in a similar way.

# Two operators on $\mathbb{V}$, cont.

**Example**

We have

$$(xxyy)y^{-1} = xxy, \qquad (xyyx)y^{-1} = 0, \qquad x^{-1}(xyxxy + yxxxy) = yxxy.$$

**Example**

We have

$$(xxyy)y^{-2} = xx, \qquad (xxyy)y^{-3} = 0, \qquad x^{-2}(xyxxy + yxxxy) = 0.$$

# Two operators on $\mathbb{V}$, cont.

Recall the Rosso embedding $U_q^+ \to U \subsetneq \mathbb{V}$.

**Theorem (Post & Terwilliger 2020)**

*We have*

$$x^{-1}U \subseteq U, \qquad Uy^{-1} \subseteq U.$$

# The $K$-matrix

## Definition

For $j \in \frac{1}{2}\mathbb{N}^+$, define the $(2j+1) \times (2j+1)$ matrix $K^{(j)}(t)$ by

$$K^{(j)}_{(a,b)}(t) = \varphi(a,b,j) t^{\frac{a-b-2j}{2}} x^{-(b-1)} \Delta^{(-2j)}(-t) y^{-(2j+1-a)},$$

where $\varphi(a,b,j)$ is a scalar whose lengthy formula is omitted.

By the theorem from the previous slide, all the coefficients in $K^{(j)}_{(a,b)}(t)$ are contained in $U$.

## Example

We have

$$K^{(\frac{1}{2})}(t) = \begin{pmatrix} qt^{\frac{1}{2}} W^-(t) & G(t) \\ \tilde{G}(t) & qt^{\frac{1}{2}} W^+(t) \end{pmatrix}.$$

# The generalized reflection equation

**Theorem (Ruan 2025)**

*For $j_1, j_2 \in \frac{1}{2}\mathbb{N}^+$,*

$$R^{(j_1,j_2)}(t/s) \star \left(K^{(j_1)}(s^2) \otimes I_{2j_2+1}\right) \star \widehat{R}^{(j_1,j_2)} \star \left(I_{2j_1+1} \otimes K^{(j_2)}(t^2)\right)$$
$$= \left(I_{2j_1+1} \otimes K^{(j_2)}(t^2)\right) \star \widehat{R}^{(j_1,j_2)} \star \left(K^{(j_1)}(s^2) \otimes I_{2j_2+1}\right) \star R^{(j_1,j_2)}(t/s).$$

The above equation is said to be of **Freidel-Maillet type**.

The case $j_1 = j_2 = \frac{1}{2}$ is the $q$-shuffle version of a result proved by **Baseilhac 2021**. We remark that this result is another motivation of the above theorem.

# Summary

In this talk, we recall the algebra $U_q^+$ and the Rosso embedding.

Using the Rosso embedding, we presented a uniform approach to three PBW basis for $U_q^+$ due to Damiani, Beck, and Terwilliger respectively.

We constructed a generalized reflection equation of the form $RK\widehat{R}K$, where the $K$-matrix is given in closed form using the uniform approach.

**Thank you for your attention!**