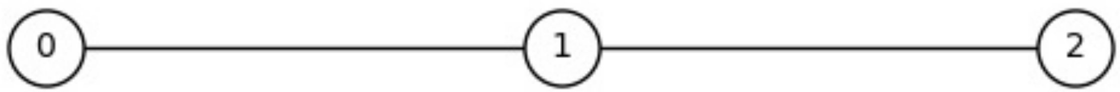
Quantum State Transfer in Association Schemes

Hanmeng (Harmony) Zhan

Worcester Polytechnic Institute

Combinatorics around the q-Onsager Algebra, June 27, 2025
TerwilligerFest, Kranjska Gora, Slovenia

Classical random walk

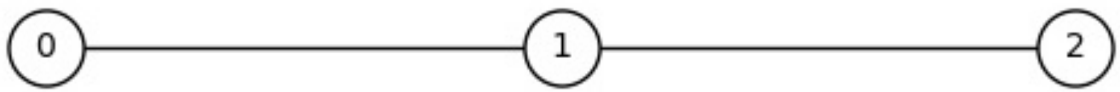


- state: probability distribution $x(t)$ in $\mathbb{R}^{V(P_3)}$
- transition matrix: stochastic adjacency matrix M of P_3

$$x(t+1) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array} x(t)$$

Classical random walk

Quantum



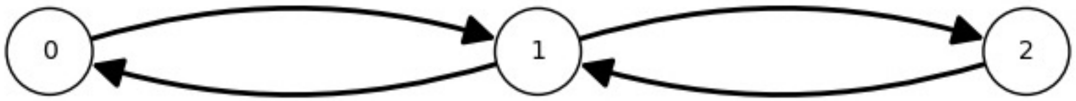
unit vector $\mathbb{C}^{V(P_3)}$

- state: probability distribution $x(t)$ in $\mathbb{R}^{V(P_3)}$
- transition matrix: stochastic adjacency matrix M of P_3

unitary

$$x(t+1) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array} x(t)$$

A discrete quantum walk on the arcs



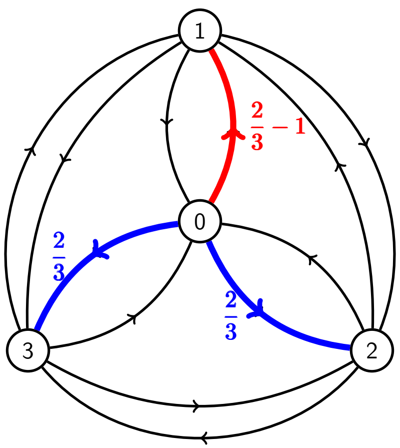
- state: unit vector $x(t)$ in $\mathbb{C}^{\mathrm{arcs}(P_3)}$
- transition matrix: unitary adjacency matrix U of the line digraph of P_3, where (a,b) is adjacent to (c,d) if $b=c$

$$x(t+1) = \begin{array}{c} \\ (0,1) \\ (1,0) \\ (1,2) \\ (2,1) \end{array} \begin{array}{c} \begin{array}{cccc} (0,1) & (1,0) & (1,2) & (2,1) \end{array} \\ \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \end{array} x(t)$$

A coined quantum walk with Grover coins

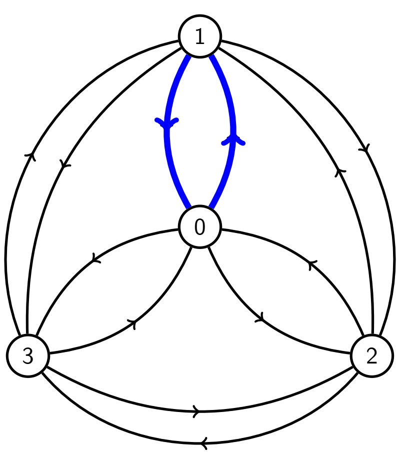
$U = RC$, where

- C is the coin matrix (changes direction but preserves the tail)



$$C = \begin{pmatrix} \frac{2}{3}J - I & & & \\ & \frac{2}{3}J - I & & \\ & & \frac{2}{3}J - I & \\ & & & \frac{2}{3}J - I \end{pmatrix}$$

- R is the arc-reversal matrix (moves the tail in that direction)



$$R : e_{(a,b)} \mapsto e_{(b,a)}$$

(Aharonov, Ambainis, Kempe, 2001)

A discrete quantum walk on X

- Initial state: unit vector $x \in \mathbb{C}^{\mathrm{arcs}(X)}$
- Transition matrix: unitary matrix $U = RC \in \mathbb{C}^{\mathrm{arcs}(X) \times \mathrm{arcs}(X)}$
- State at time t: $U^t x$
- Outcome of measurement: an arc (a, b), with probability

$$p_t(x \to (a,b)) := ((U^t x) \circ \overline{(U^t x)})_{(a,b)}$$

- Probability at a vertex a:

$$p_t(x \to a) := \sum_{b \sim a} p_t(x \to (a,b))$$

Perfect state transfer

Perfect state transfer from a to b:

$$\exists t : \left|\left\langle U^t x_a, x_b \right\rangle\right| = 1,$$

where x_a and x_b are unit vectors with

$$\operatorname{supp}(x_a) = \{(a,u) : u \sim a\}, \quad \operatorname{supp}(x_b) = \{(b,v) : v \sim b\}.$$

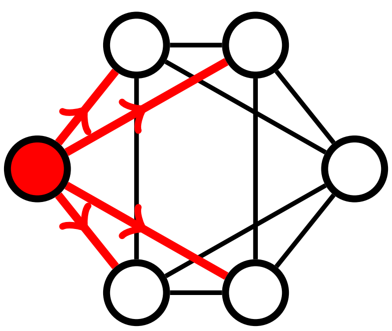


Figure: At time 0

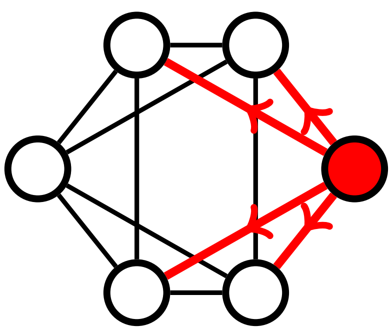


Figure: At time t

Pretty good state transfer

Pretty good state transfer from a to b:

$$\forall \epsilon > 0, \exists t : \left|\left\langle U^t x_a, x_b \right\rangle\right| > 1 - \epsilon,$$

where x_a and x_b are unit vectors with

$$\operatorname{supp}(x_a) = \{(a,u) : u \sim a\}, \quad \operatorname{supp}(x_b) = \{(b,v) : v \sim b\}.$$

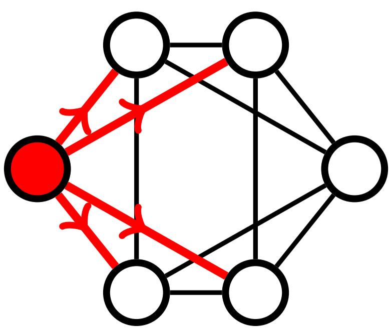


Figure: At time 0

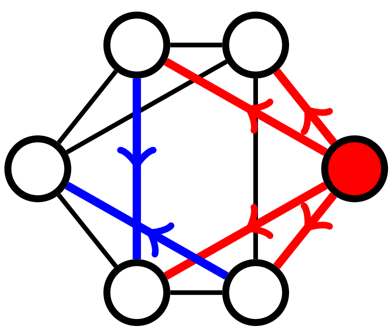


Figure: At time t

Local uniform mixing

Local uniform mixing at a:

$$\exists t : (U^t x_a) \circ (\overline{U^t x_a}) = \frac{1}{|\mathrm{arcs}(X)|}\mathbf{1},$$

where x_a is a unit vector with

$$\mathrm{supp}(x_a) = \{(a, u) : u \sim a\}.$$

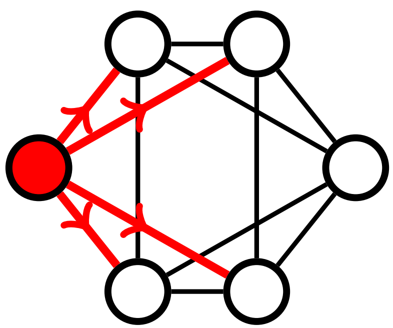


Figure: At time 0

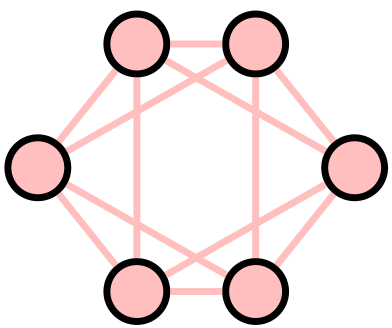


Figure: At time t

Local ϵ-uniform mixing

Local ϵ-uniform mixing at a :

$$\forall \epsilon > 0, \exists t : \left| \left\langle (U^t x_a) \circ (\overline{U^t x_a}), \frac{1}{|\mathrm{arcs}(X)|} \mathbf{1} \right\rangle \right| > 1 - \epsilon,$$

where x_a is a unit vector with

$$\mathrm{supp}(x_a) = \{(a, u) : u \sim a\}.$$

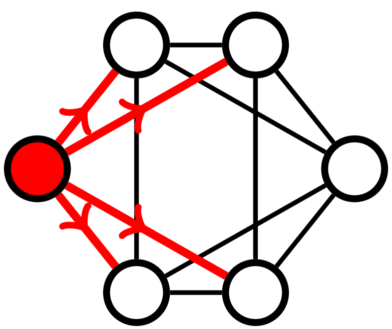


Figure: At time 0

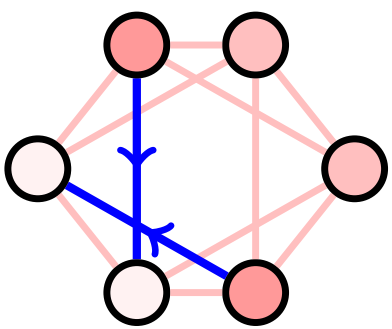


Figure: At time t

From coins to weighted adjacency matrices

Diagonalizing U

If $U = RC$

then

$\mathbb{C}^{\mathrm{arcs}(X)}$ — state space

$W_1 \oplus W_2 \oplus \cdots$ — $\langle R, C \rangle$-invariant subspaces

$\mathrm{span}\{z_1\} \oplus \mathrm{span}\{z_2\} \oplus \mathrm{span}\{z_3\} \oplus \cdots$ — 1-dim U-invariant subspaces

Diagonalizing U

If $U = RC$ where R and C are reflections:

$$R^2 = C^2 = I, \quad R^* = R, \quad C^* = C$$

then

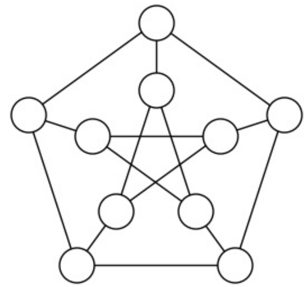
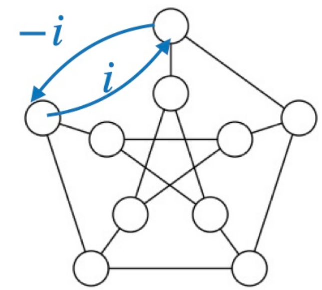
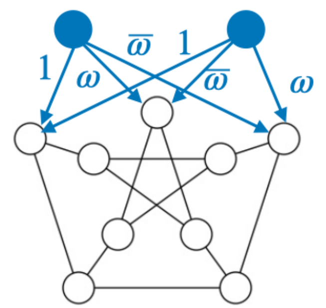
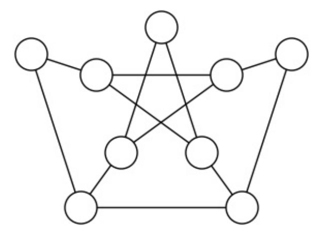
$\mathbb{C}^{\mathrm{arcs}(X)}$ state space

$W_1 \oplus W_2 \oplus \cdots$ $\langle R, C\rangle$-invariant subspaces of dim 1 or 2

span$\{z_1\}\oplus$span$\{z_2\}\oplus$span$\{z_3\}\oplus \cdots$ 1-dim U-invariant subspaces

The spectral decomposition of U is determined by those of R, C and CRC (Godsil and Zhan, 2023)

Coins vs weights

coin matrix	Hermitian adjacency matrix
$\begin{pmatrix} \frac{2}{k}J-I & & \\ & \frac{2}{k}J-I & \\ & & \ddots \end{pmatrix}$	
$\begin{pmatrix} \frac{2}{k}\begin{pmatrix} 1 & i & i \\ -i & 1 & 1 \\ -i & 1 & 1 \end{pmatrix}-I & & \\ & \frac{2}{k}J-I & \\ & & \ddots \end{pmatrix}$	
$\begin{pmatrix} I-\frac{2}{k}J & & \\ & \frac{2}{k}J-I & \\ & & \ddots \end{pmatrix}$	
$\begin{pmatrix} -I & & \\ & \frac{2}{k}J-I & \\ & & \ddots \end{pmatrix}$	

Pretty good state transfer in schemes

Assuming $Cx_a = x_a$ and $Cx_b = x_b$

Eigenvalue support and strong cospectrality

Let H of a Hermitian adjacency matrix of X with spectral decompoistion

$$H = \sum_{\lambda} \lambda E_{\lambda}.$$

The eigenvalue support of a is

$$\Lambda_a = \{\lambda : E_{\lambda} e_a \neq 0\}.$$

Eigenvalue support and strong cospectrality

Let H of a Hermitian adjacency matrix of X with spectral decompoistion

$$H = \sum_{\lambda} \lambda E_{\lambda}.$$

The eigenvalue support of a is

$$\Lambda_a = \{\lambda : E_{\lambda} e_a \neq 0\}.$$

a and b are strongly cospectral if for each λ:

$$E_{\lambda} e_a \parallel E_{\lambda} e_b \quad \text{and} \quad \|E_{\lambda} e_a\| = \|E_{\lambda} e_b\|.$$

Eigenvalue support and strong cospectrality

Let H of a Hermitian adjacency matrix of X with spectral decompoistion

$$H = \sum_{\lambda} \lambda E_{\lambda}.$$

The eigenvalue support of a is

$$\Lambda_a = \{\lambda : E_{\lambda} e_a \neq 0\}.$$

a and b are strongly cospectral if for each λ:

$$E_{\lambda} e_a \parallel E_{\lambda} e_b \quad \text{and} \quad \|E_{\lambda} e_a\| = \|E_{\lambda} e_b\|.$$

Strongly cospectral vertices have the same eigenvalue support.

Characterizing PGST

Theorem (Chan and Zhan, 2023; Zhan, 2025+)

Let H be a normalized Hermitian adjacency matrix of a graph X with

$$H = \sum_{\lambda} \lambda E_{\lambda},$$

A quantum walk associated with H admits ab-PGST if and only if

1. *a and b are γ-strongly cospectral for some unimodular $\gamma \in \mathbb{C}$:*

$$E_{\lambda} e_a = \pm \gamma E_{\lambda} e_b, \quad \forall \lambda$$

2. *For any set $\{\ell_{\lambda} : \lambda \in \Lambda_a\}$ of integers such that*

$$\sum_{\lambda \in \Lambda_a} \ell_{\lambda} \arccos \lambda \equiv 0 \pmod{2\pi},$$

   we have

$$\sum_{\lambda \in \Lambda_{ab}^{-}} \ell_{\lambda} \equiv 0 \pmod{2}.$$

An example with Grover coins

PGST happens on the hypercube Q_5 between antipodal vertices

1. The antipodal vertices of Q_5 are strongly cospectral, with

$$\Lambda^+ = \left\{1, \frac{1}{5}, -\frac{3}{5}\right\}, \quad \Lambda^- = \left\{\frac{3}{5}, -\frac{1}{5}, -1\right\}.$$

2. The following implication is true:

$$\ell_1 \arccos(1) + \ell_{\frac{1}{5}} \arccos\left(\frac{1}{5}\right) + \ell_{-\frac{3}{5}} \arccos\left(-\frac{3}{5}\right)$$
$$+\ell_{-1} \arccos(-1) + \ell_{-\frac{1}{5}} \arccos\left(-\frac{1}{5}\right) + \ell_{\frac{3}{5}} \arccos\left(\frac{3}{5}\right) \equiv_{2\pi} 0$$

$$\implies \ell_{-1} + \ell_{-\frac{1}{5}} + \ell_{\frac{3}{5}} \equiv_2 0$$

Hypercubes

- Between $\mathbf{0}$ and $\mathbf{1}$ on Q_d
  - d prime; unweighted (Chan and Zhan, 2023)

Hypercubes

- Between $\mathbf{0}$ and $\mathbf{1}$ on Q_d
  - d prime; unweighted (Chan and Zhan, 2023)
  - d composite: ?

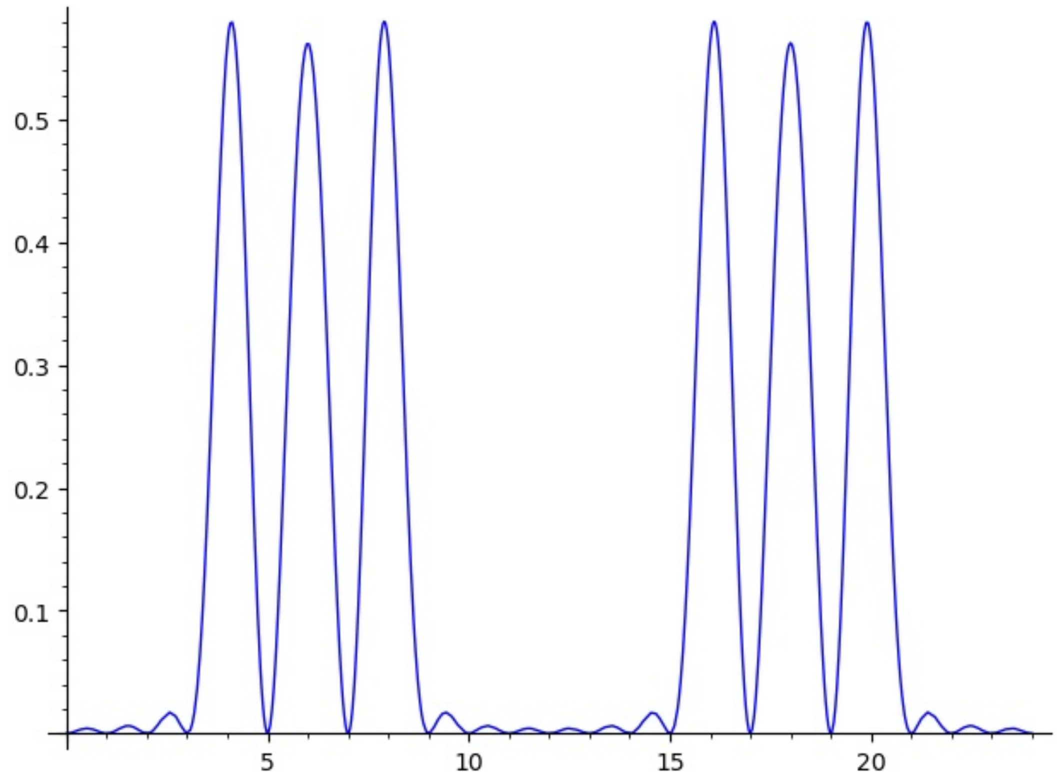


Figure: Q_4: transfer probability between antipodal vertices with Grover coins

A sufficient condition on graph spectra

Theorem (Zhan, 2025+)

Let H be a real symmetric adjacency matrix of X. A quantum walk associated with H admits ab-PGST if the following hold.

1. *The spectral radius of H is a prime p.*
2. *a and b are strongly cospectral relative to H, and*

$$\Lambda_a \subseteq \{p - 2r : r = 0, 1, \cdots, p\}.$$

3. *The eigenvalue support Λ_a satisfies one of the following.*
    - *For any pair $\lambda, -\lambda$ in Λ_a, we have $\lambda \in \Lambda_{ab}^{\pm} \iff -\lambda \in \Lambda_{ab}^{\pm}$.*
    - *For any pair $\lambda, -\lambda$ in Λ_a, we have $\lambda \in \Lambda_{ab}^{\pm} \iff -\lambda \in \Lambda_{ab}^{\mp}$.*

Proof. The angles

$$\{\pi\} \cup \{\arccos(\lambda/p) : \lambda \in \Lambda_a, 0 < \lambda < p\}$$

are linearly independent over $\mathbb{Q}$. □

Enabling PGST with weighted Grover coins

- Between $\mathbf{0}$ and $\mathbf{1}$ on Q_d
  - d prime; unweighted (Chan and Zhan, 2023)
  - d composite: weighted with 2 weights (Zhan, 2025+)

Enabling PGST with weighted Grover coins

- Between $\mathbf{0}$ and $\mathbf{1}$ on Q_d
  - d prime; unweighted (Chan and Zhan, 2023)
  - d composite: weighted with 2 weights (Zhan, 2025+)
- Between $\mathbf{0}$ and $\mathbf{1}$ on folded d-cube weighted
  - any d: weighted with 2 weights (Martin, O'Toole and Zhan, 2025++)

Enabling PGST with weighted Grover coins

- Between $\mathbf{0}$ and $\mathbf{1}$ on Q_d
  - d prime; unweighted (Chan and Zhan, 2023)
  - d composite: weighted with 2 weights (Zhan, 2025+)
- Between $\mathbf{0}$ and $\mathbf{1}$ on folded d-cube weighted
  - any d: weighted with 2 weights (Martin, O'Toole and Zhan, 2025++)
- Between $\mathbf{0}$ and b on connected cubelike graph
  - any connection set: weighted with at most 3 weights

PGST on connected $X(\mathbb{Z}_2^d, C)$

Proof. If $X = Q_d$, suppose $b \neq \mathbf{1}$. Then $C = S \sqcup \{c\} \sqcup T$ with $b = \sum_{s \in S} s$. Let

$$H = w \sum_{s \in S} A_s + m A_c + \sum_{t \in T} A_t,$$

$\rho(H)$

where $m \geq 1$, $w \geq 2(m + |T|)$ and $w|S| + m + |T|$ is an odd prime.

- $\lambda_g \pm \lambda_h = w(\psi_g(S) \pm \psi_h(S)) + m(\psi_g(c) \pm \psi_h(c)) + \psi_g(T) \pm \psi_h(T)$
- $\lambda_g - \lambda_h$ is even

$\psi_g(x) = (-1)^{\langle g, x \rangle}$

- $\dfrac{\psi_g(b)}{\psi_h(b)} = \prod_{s \in S} (-1)^{\langle g,s \rangle} (-1)^{\langle h,s \rangle} = (-1)^{\left|S \cap g^\perp\right| + \left|S \cap h^\perp\right|}$
- If $\lambda_g = \lambda_h$, then $\psi_g(S) = \psi_h(S)$, so $\left|S \cap g^\perp\right| = \left|S \cap h^\perp\right|$.
- If $\lambda_h = -\lambda_h$, then $\psi_g(S) = -\psi_h(S)$, so $\left|S \cap g^\perp\right| + \left|S \cap h^\perp\right| = |S|$.

Problems on PGST

1. Which graphs in translation schemes admit PGST with $H \in \mathbb{C}(\mathcal{A})$?

   Can we determine the minimum number of distinct weights needed?

2. For which schemes do we need complex Hermitian H for PGST?

3. Study strongly cospectral vertices relative to a unimodular $\gamma \in \mathbb{C}$:

$$E_\lambda e_a = \pm\gamma E_\lambda e_b, \quad \forall\lambda$$

ϵ-uniform mixing in schemes

Assuming $Cx_a = x_a$

Connection to generalized PGST

Perfect state transfer from x to y:

$$\exists t : \left|\left\langle U^t x, y \right\rangle\right| = 1.$$

Pretty good state transfer from x to y:

$$\forall \epsilon > 0, \exists t : \left|\left\langle U^t x, y \right\rangle\right| > 1 - \epsilon,$$

Connection to generalized PGST

Perfect state transfer from x to y:

$$\exists t : \left| \left\langle U^t x, y \right\rangle \right| = 1.$$

Pretty good state transfer from x to y:

$$\forall \epsilon > 0, \exists t : \left| \left\langle U^t x, y \right\rangle \right| > 1 - \epsilon,$$

A state x is flat if all entries of x have the same absolute value.

Connection to generalized PGST

Perfect state transfer from x to y:

$$\exists t : \left|\left\langle U^t x, y \right\rangle\right| = 1.$$

Pretty good state transfer from x to y:

$$\forall \epsilon > 0, \exists t : \left|\left\langle U^t x, y \right\rangle\right| > 1 - \epsilon,$$

A state x is flat if all entries of x have the same absolute value.

- local uniform mixing at $a \iff$ PST from x_a to a flat state
- local ϵ-uniform mixing at $a \iff$ PGST from x_a to a flat state

Generalized PGST

Theorem (Chan and Zhan, 2023)

Let the spectral decomposition of U be

$$U = \sum_{\theta} e^{i\theta} F_{\theta}.$$

Then U admits xy-PGST if and only if the following hold.

1. *x and y are strongly cospectral relative to U:*

$$F_{\theta} x = e^{i\delta} F_{\theta} y, \quad \forall \theta$$

2. *There is a unimodular $\gamma \in \mathbb{C}$ such that, for any $\epsilon > 0$, there exists $t \in \mathbb{R}$ such that if $F_{\theta} x \neq 0$, then $\left| e^{i(t\theta+\delta)} - \gamma \right| < \epsilon$.*

Grover coins

Theorem (Zhan, 2025+)

Let X be a regular graph that admits local ϵ-uniform mixing with Grover coins.

1. *If X is non-bipartite with local ϵ-uniform mixing, then n is a perfect square.*
2. *If, in addition, X lies in a scheme, then X admits ϵ-uniform mixing.*
3. *If, in addition, X is complete or strongly regular, then the adjacency algebra of X contains a real Hadamard matrix.*

Grover coins

Theorem (Zhan, 2025+)

Let X be a regular graph that admits local ϵ-uniform mixing with Grover coins.

1. *If X is non-bipartite with local ϵ-uniform mixing, then n is a perfect square.*
2. *If, in addition, X lies in a scheme, then X admits ϵ-uniform mixing.*
3. *If, in addition, X is complete or strongly regular, then the adjacency algebra of X contains a real Hadamard matrix.*

Theorem (Zhan, 2025+)

A strongly regular graph X admits ϵ-uniform mixing if and only if X or $\overline{X}$ has parameters $(4m^2, 2m^2 \pm m, m^2 \pm m, m^2 \pm m)$ where $m \geq 2$.

Restricting the target state

X distance regular with distance partition $\{C_0, \cdots, C_d\}$ relative to a.

Restricting the target state

X distance regular with distance partition $\{C_0, \cdots, C_d\}$ relative to a.

- The partition $\{C_{ij}\}$ where

$$C_{ij} = \{(u, v) : u \sim v, \mathrm{dist}(a, u) = i, \mathrm{dist}(a, v) = j\}$$

  is an equitable partition of the line digraph of X

Restricting the target state

X distance regular with distance partition $\{C_0, \cdots, C_d\}$ relative to a.

- The partition $\{C_{ij}\}$ where

$$C_{ij} = \{(u,v) : u \sim v, \mathrm{dist}(a,u) = i, \mathrm{dist}(a,v) = j\}$$

  is an equitable partition of the line digraph of X

- If S is the characteristic matrix of the above parititon, then $\mathrm{col}(S)$ is U-invariant, and $U^t x_a \in \mathrm{col}(S)$ for any $t \in \mathbb{R}$

Restricting the target state

X distance regular with distance partition $\{C_0, \cdots, C_d\}$ relative to a.

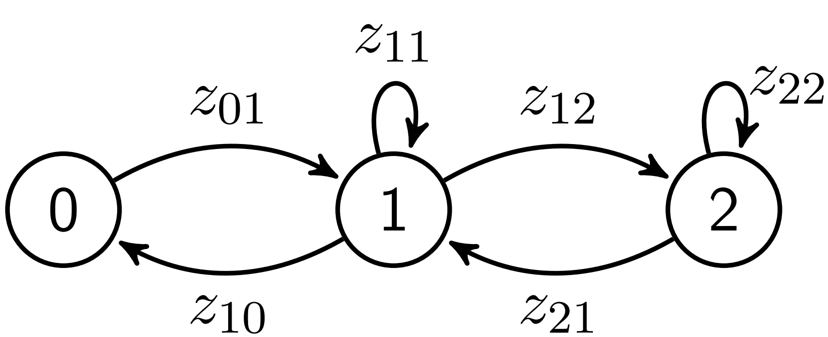
- The partition $\{C_{ij}\}$ where

$$C_{ij} = \{(u, v) : u \sim v, \mathrm{dist}(a, u) = i, \mathrm{dist}(a, v) = j\}$$

  is an equitable partition of the line digraph of X

- If S is the characteristic matrix of the above parititon, then $\mathrm{col}(S)$ is U-invariant, and $U^t x_a \in \mathrm{col}(S)$ for any $t \in \mathbb{R}$

- If there is PGST from x_a to y, then $y \in \mathrm{col}(S)$



Restricting the target state

X distance regular with distance partition $\{C_0, \cdots, C_d\}$ relative to a.

- The partition $\{C_{ij}\}$ where

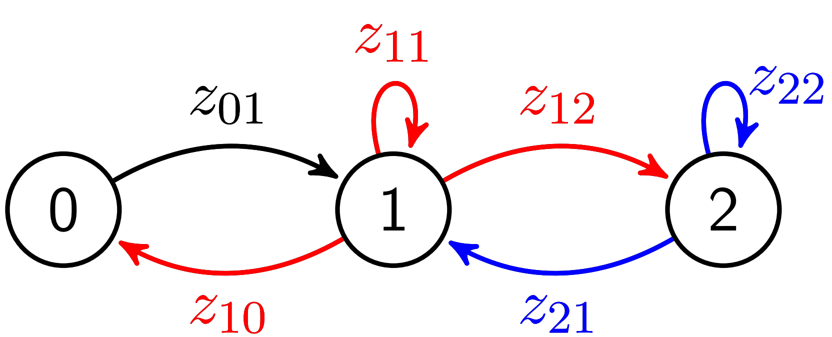
$$C_{ij} = \{(u, v) : u \sim v, \mathrm{dist}(a, u) = i, \mathrm{dist}(a, v) = j\}$$

  is an equitable partition of the line digraph of X

- If S is the characteristic matrix of the above parititon, then $\mathrm{col}(S)$ is U-invariant, and $U^t x_a \in \mathrm{col}(S)$ for any $t \in \mathbb{R}$

- If there is PGST from x_a to y, then $y \in \mathrm{col}(S)$

- If X is strongly regular, then y or Ry is constant on outgoing arcs of the same vertex



DRG: consequences of strong cospectrality

Let P be the eigenmatrix of the scheme. Let $y = Sz$. There are scalars δ_r where $\delta_0 \in \{0, \pi\}$ such that

$$\sqrt{k}\begin{pmatrix}\cos\delta_0\\ \cos\delta_1\\ \vdots\\ \cos\delta_d\end{pmatrix} = P\begin{pmatrix}b_0 & & & & & \\ & c_1 & a_1 & b_1 & & \\ & & & & \ddots & \\ & & & & c_d & a_d\end{pmatrix} z$$

and

$$\sqrt{k}\begin{pmatrix}\cos(\delta_0+\theta_0)\\ \cos(\delta_1+\theta_1)\\ \vdots\\ \cos(\delta_d+\theta_d)\end{pmatrix} = P\begin{pmatrix}b_0 & & & & & \\ & c_1 & a_1 & b_1 & & \\ & & & & \ddots & \\ & & & & c_d & a_d\end{pmatrix} R'z$$

Problems on ϵ-uniform mixing

Let X be a graph in a scheme.

1. If X is non-bipartite distance regular graph, and x_a is strongly cospectral to a flat vector y, must y or Ry be constant on outgoing arcs of the same vertex?

2. What about bipartite graphs?

   $U^t x_a$ has flat imaginary parts

3. Do weighted Grover coins enable ϵ-uniform mixing?

   Complex Hadamard matrix?

4. What about other target probability distributions?

   Strong cospectrality between x_a and a general state

Thank you!