Homomorphisms from the tetrahedron algebra $\boxtimes$ to the special orthogonal algebra $\mathfrak{so}_4$

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Lie algebras

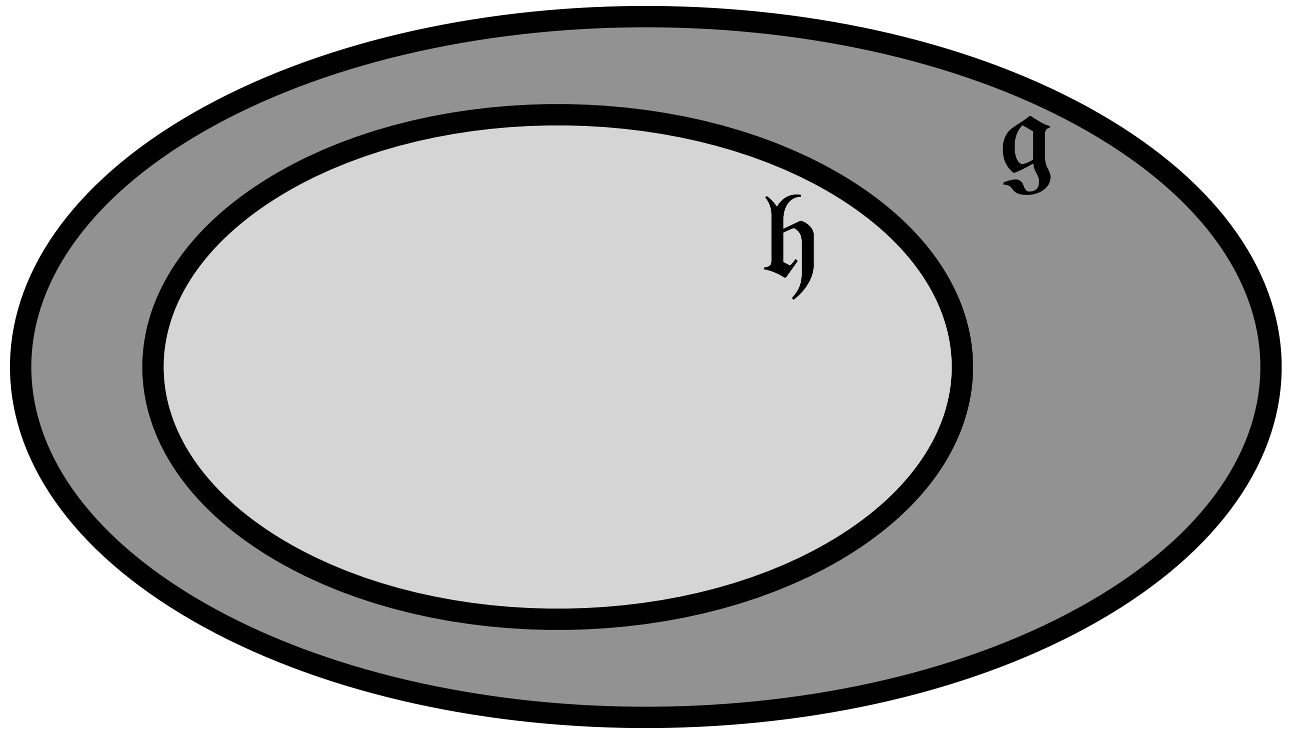
Throughout, let $\mathbb{F}$ denote an algebraically closed field with characteristic 0.

By a **Lie algebra** over $\mathbb{F}$, we mean a vector space $\mathfrak{g}$ over $\mathbb{F}$ together with a Lie bracket operation $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (i)-(iii) below.

(i) $[\,,\,]$ is bilinear.

(ii) $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.

(iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.



The special orthogonal algebra $\mathfrak{so}_4$

The **special orthogonal algebra** $\mathfrak{so}_4$ is the Lie algebra over $\mathbb{F}$ with Chevalley basis $\{e_1, h_1, f_1, e_2, h_2, f_2\}$ satisfying the relations

$$
\begin{aligned}
&[h_i, e_j] = 2\delta_{ij}e_i, && (i,j \in \{1,2\}),\\
&[f_i, h_j] = 2\delta_{ij}f_i, && (i,j \in \{1,2\}),\\
&[e_i, f_j] = \delta_{ij}h_i, && (i,j \in \{1,2\}),\\
&[e_i, e_j] = 0, [h_i, h_j] = 0, [f_i, f_j] = 0, && (i,j \in \{1,2\}),
\end{aligned}
$$

where δ is the Kronecker delta function.

there is an automorphism $* : \mathfrak{so}_4 \to \mathfrak{so}_4$ with parameters $p_1, p_2 \in \mathbb{F}$ such that $p_1, p_2 \notin \{0,1\}$ ⇒ $\{e_1^*, h_1^*, f_1^*, e_2^*, h_2^*, f_2^*\}$ is another Chevalley basis of $\mathfrak{so}_4$

For convenience, we define the elements $h^{\downarrow}, h^{\uparrow} \in \mathfrak{so}_4$ by

$$
h^a = \begin{cases} h_1 + h_2, & \text{if } a = \downarrow \\ h_1 - h_2, & \text{if } a = \uparrow \end{cases}, \qquad (a \in \{\downarrow, \uparrow\}),
$$

and define the elements $h^{*\downarrow}, h^{*\uparrow} \in \mathfrak{so}_4$ analogously.

The tetrahedron algebra ⊠

Let ⊠ denote the Lie algebra over $\mathbb{F}$ with generators $\{x_{ij} \mid i, j \in \{0,1,2,3\}, i \neq j\}$ subject to the following relations:

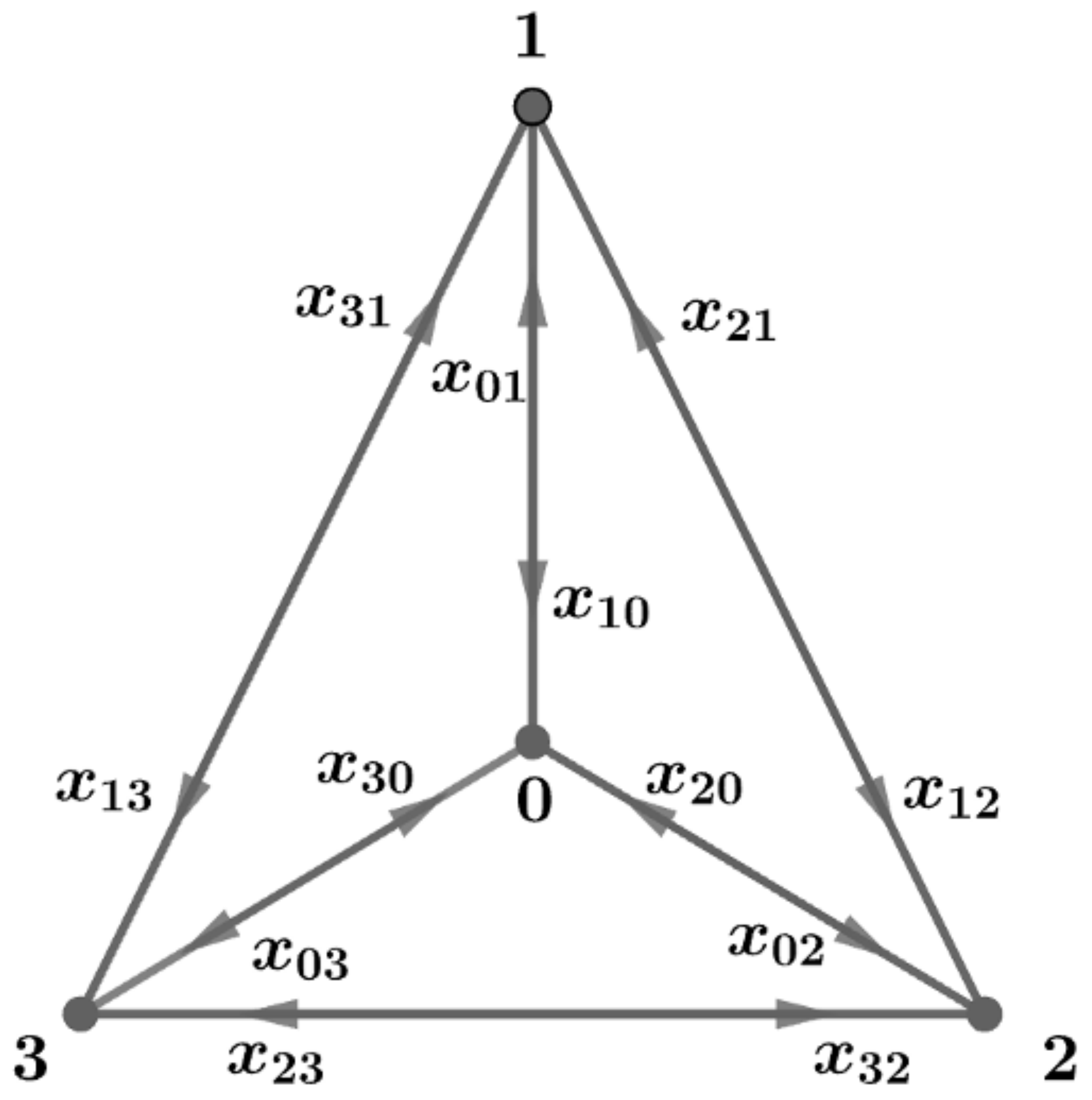
(i) For distinct $i, j \in \{0,1,2,3\}$, we have $x_{ij} + x_{ji} = 0$.

(ii) For mutually distinct $h, i, j \in \{0,1,2,3\}$, we have

$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}.$$

(iii) For mutually distinct $h, i, j, k \in \{0,1,2,3\}$, we have

$$[x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4[x_{hi}, x_{jk}].$$



Finite-dimensional modules of Lie algebras

Let W denote a finite-dimensional vector space over $\mathbb{F}$.

Let $\text{End}(W)$ denote the set of all $\mathbb{F}$-linear maps $W \to W$.

Let $\mathfrak{g}$ denote a Lie algebra over $\mathbb{F}$.

The pair (π, W) is called a **$\mathfrak{g}$-module** if there is a linear map $\pi : \mathfrak{g} \to \text{End}(W)$ such that

$$\pi([x, y]) = [\pi(x), \pi(y)], \qquad (x, y \in \mathfrak{g}) \,.$$

We say the $\mathfrak{g}$-module (π, W) is **irreducible** whenever there is no nontrivial subspace $W' \subseteq W$ such that (π, W') is a $\mathfrak{g}$-module.

The finite-dimensional irreducible $\mathfrak{so}_4$-module (π, V)

Let y_1, z_1, y_2, z_2 denote mutually commuting indeterminates. For integers $d, p \geq 1$, let

$$\mathbb{I} = \{(i,j) \mid i = 0,1,\ldots,d; j = 0,1,\ldots,p\},$$

and define the $\mathbb{F}$-vector space $V = V(d,p)$ with basis consisting of

$$v_{ij} := y_1^{d-i} z_1^{i} y_2^{p-j} z_2^{j}, \qquad ((i,j) \in \mathbb{I}).$$

The pair (π, V) is a finite-dimensional irreducible $\mathfrak{so}_4$-module via the linear map $\pi : \mathfrak{so}_4 \to \mathrm{End}(V)$ such that

$$e_i \mapsto y_i \frac{\partial}{\partial z_i}, \qquad (i \in \{1,2\}),$$

$$h_i \mapsto y_i \frac{\partial}{\partial y_i} - z_i \frac{\partial}{\partial z_i}, \qquad (i \in \{1,2\}),$$

$$f_i \mapsto z_i \frac{\partial}{\partial y_i}, \qquad (i \in \{1,2\}),$$

where $\dfrac{\partial}{\partial t}$ means partial derivative with respect to t.

Tridiagonal pairs

Let W denote a finite-dimensional vector space over $\mathbb{F}$. By a **tridiagonal pair** (or **TD pair**) on W, we mean an ordered pair (A, A^*) of linear maps $A : W \to W$ and $A^* : W \to W$ satisfying (i)-(iv) below.

i. Each of A, A^* is diagonalizable on W.

ii. There exists an ordering $W_0, W_1, \ldots, W_D$ of eigenspaces of A such that

$$A^* W_i \subseteq W_{i-1} + W_i + W_{i+1}, \qquad (i \in \{0,1,\ldots,D\}),$$

where $W_{-1} = 0$ and $W_{D+1} = 0$.

iii. There exists an ordering $W^*_0, W^*_1, \ldots, W^*_\Delta$ of eigenspaces of A^* such that

$$A W^*_i \subseteq W^*_{i-1} + W^*_i + W^*_{i+1}, \qquad (i \in \{0,1,\ldots,\Delta\}),$$

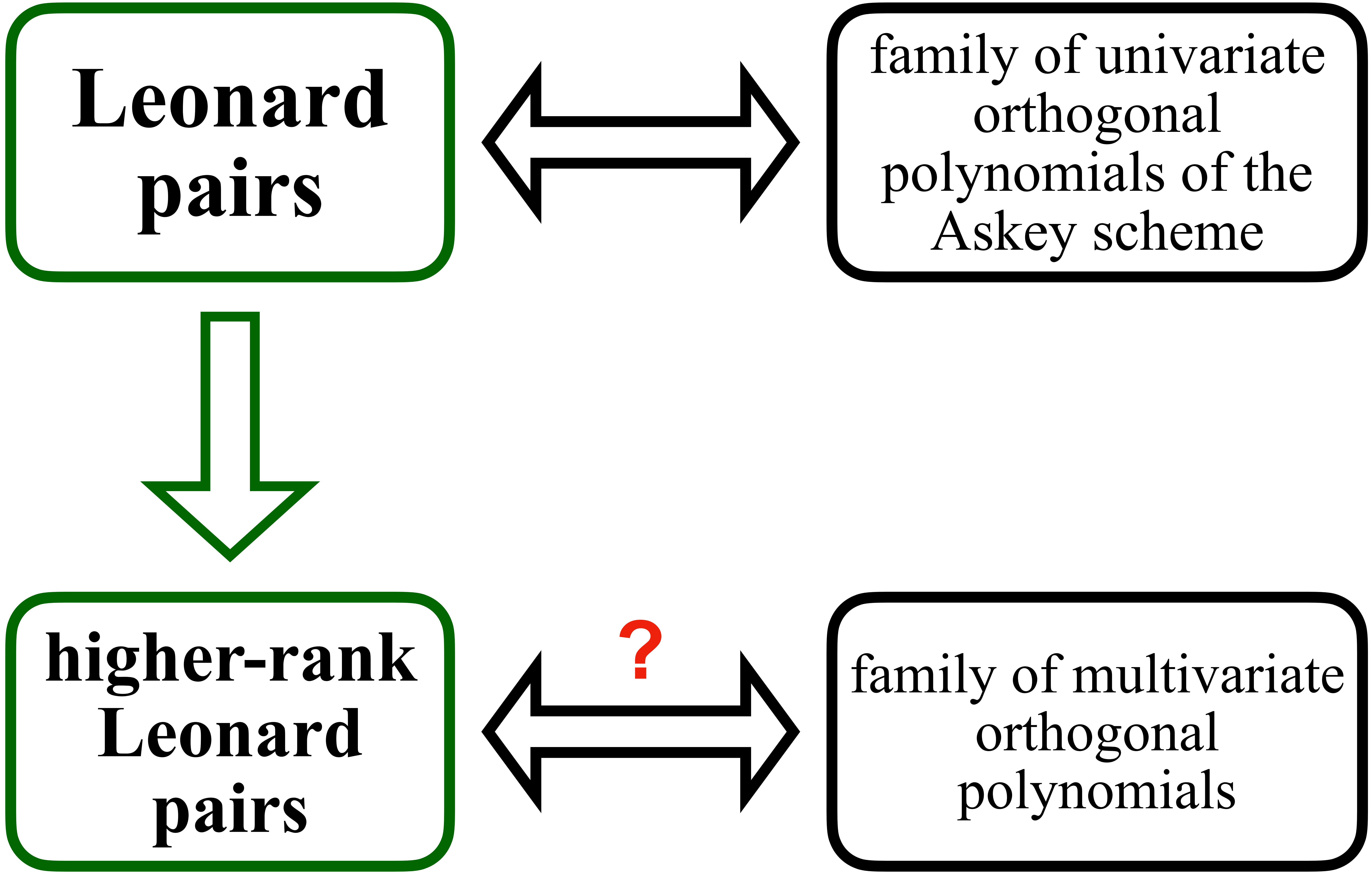
where $W^*_{-1} = 0$ and $W_{\Delta+1} = 0$.

iv. There is no nontrivial subspace W' of W such that $AW' \subseteq W'$ and $A^*W' \subseteq W'$.

$\{W_i\}_{i=0}^D$, $\{W^*_i\}_{i=0}^D$ ↔ $\{D-2i\}_{i=0}^D$ ⟹ (A, A^*) has the **Krawtchouk type**

$\dim W_i = 1$, $\dim W^*_i = 1$ ⟹ (A, A^*) is a **Leonard pair**

Motivation of the research



Tridiagonal pairs of Krawtchouk type on V

restriction	**acts on V as a TD pair of Krawtchouk type**
$p_1 \neq p_2$	$(\pi(h^{\downarrow}), \pi(h^{*\downarrow}))$ $(\pi(h^{\uparrow}), \pi(h^{*\uparrow}))$
$p_1 \neq 1 - p_2$	$(\pi(h^{\downarrow}), \pi(h^{*\uparrow}))$ $(\pi(h^{\uparrow}), \pi(h^{*\downarrow}))$

The Lie algebra homomorphism $\tau_{ab} : \boxtimes \to \mathfrak{so}_4$

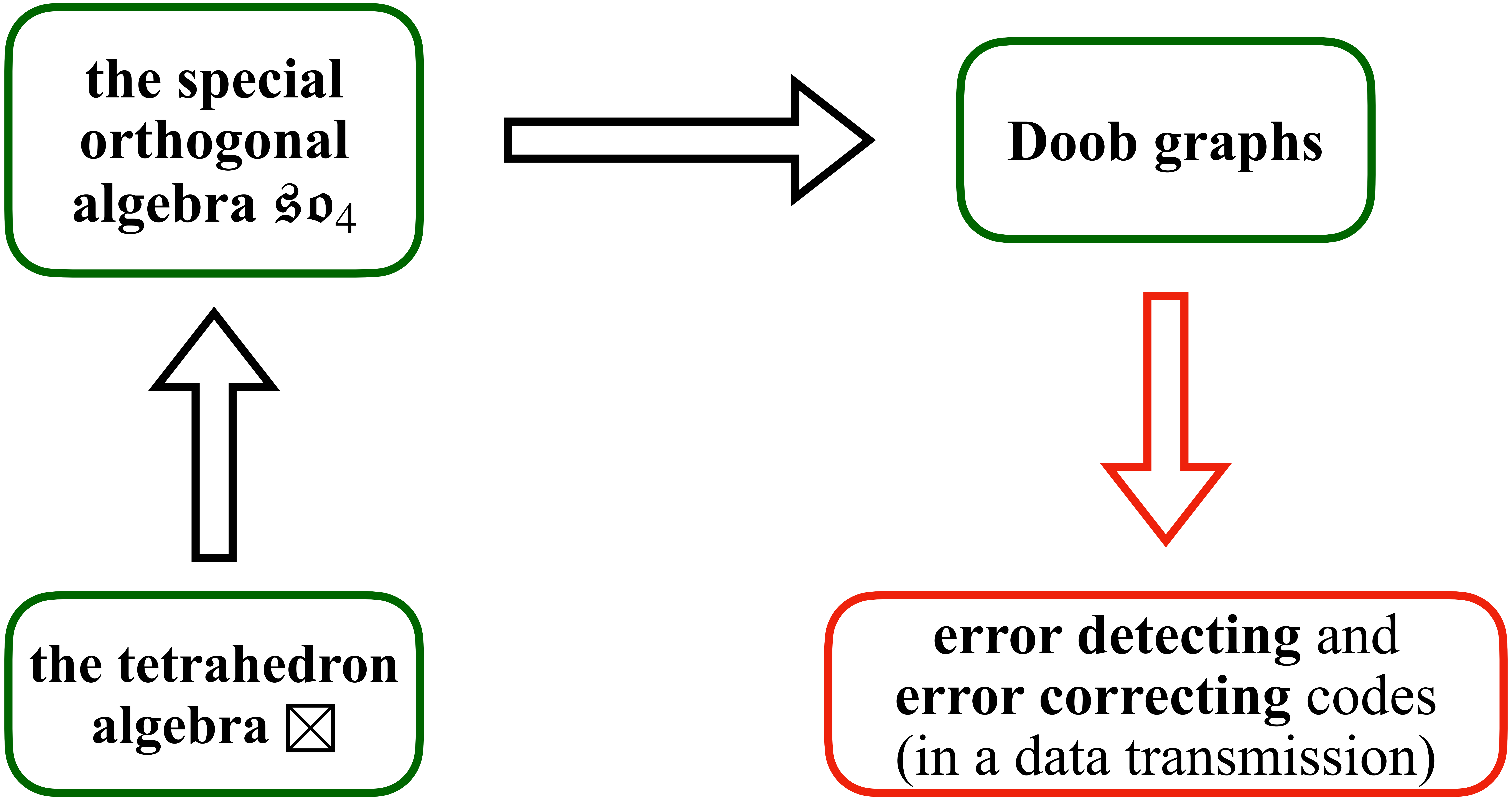
For $a, b \in \{ \downarrow, \uparrow \}$, there is a Lie algebra homomorphism $\tau_{ab} : \boxtimes \to \mathfrak{so}_4$ such that

$$\begin{aligned} x_{01} &\mapsto h^a, \\ x_{23} &\mapsto h^{*b}, \\ x_{03} &\mapsto h^{*b} + 2u^{*ab}, \\ x_{21} &\mapsto h^a + 2s^{ab}, \\ x_{31} &\mapsto h^a - 2x^{ab}, \\ x_{20} &\mapsto h^{*b} - 2r^{*ab}, \end{aligned}$$

where

a	b	u^{*ab} *equals*	s^{ab} *equals*	x^{ab} *equals*	r^{*ab} *equals*
$\downarrow$	$\downarrow$	$e_1^* + e_2^*$	$e_1 + e_2$	$\frac{1-p_1}{p_1}e_1 + \frac{1-p_2}{p_2}e_2$	$f_1^* + f_2^*$
$\uparrow$	$\uparrow$	$e_1^* - \frac{p_2}{1-p_2}f_2^*$	$e_1 - \frac{p_2}{1-p_2}f_2$	$\frac{1-p_1}{p_1}e_1 - f_2$	$f_1^* - \frac{1-p_2}{p_2}e_2^*$
$\downarrow$	$\uparrow$	$e_1^* + f_2^*$	$e_1 - \frac{1-p_2}{p_2}e_2$	$\frac{1-p_1}{p_1}e_1 - e_2$	$f_1^* + e_2^*$
$\uparrow$	$\downarrow$	$e_1^* - \frac{1-p_2}{p_2}e_2^*$	$e_1 + f_2$	$\frac{1-p_1}{p_1}e_1 + \frac{p_2}{1-p_2}f_2$	$f_1^* - \frac{p_2}{1-p_2}f_2^*$

Some applications



The finite-dimensional irreducible $\boxtimes$-module $(\pi\tau_{ab}, V)$

Let $p_1, p_2 \in \mathbb{F}$ such that $p_1, p_2 \notin \{0,1\}$ and let $a, b \in \{\downarrow, \uparrow\}$. If $p_1 \neq p_2$ and $p_1 \neq 1 - p_2$, then $(\pi\tau_{ab}, V)$ is an irreducible $\boxtimes$-module.

Moreover, the generators of $\boxtimes$ act on V as follows. For $k \in \{0, \ldots, d+p\}$ and for $a, b \in \{\downarrow, \uparrow\}$, we have

r	s	the eigenspace of $\pi\tau_{ab}(x_{rs})$ on V with eigenvalue $2k-d-p$ is
0	1	V^a_{d+p-k}
2	3	V^{*b}_{d+p-k}
0	3	$(V^a_{d+p-k} + \cdots + V^a_{d+p}) \cap (V^{*b}_0 + \cdots + V^{*b}_{d+p-k})$
2	1	$(V^{*b}_{d+p-k} + \cdots + V^{*b}_{d+p}) \cap (V^a_0 + \cdots + V^a_{d+p-k})$
3	1	$(V^{*b}_0 + \cdots + V^{*b}_k) \cap (V^a_0 + \cdots + V^a_{d+p-k})$
2	0	$(V^{*b}_{d+p-k} + \cdots + V^{*b}_{d+p}) \cap (V^a_k + \cdots + V^a_{d+p})$

The Drinfel'd polynomial $P_{(\pi\tau_{ab},V)}(t)$

Let $p_1, p_2 \in \mathbb{F}$ such that $p_1, p_2 \notin \{0,1\}$ and let $a, b \in \{ \downarrow , \uparrow \}$.

Assume $p_1 \neq p_2$ and $p_1 \neq 1 - p_2$.

The Drinfel'd polynomial associated to the irreducible $\boxtimes$-module $(\pi\tau_{ab}, V)$ is

$$P_{(\pi\tau_{ab},V)}(t) = \begin{cases} (1-(1-p_1)t)^d(1-(1-p_2)t)^p & \text{if } a = b, \\ (1-(1-p_1)t)^d(1-p_2t)^p & \text{if } a \neq b\,. \end{cases}$$

Remarks on the isomorphism of the TD pairs on V

restrictions	**isomorphic TD pairs on V**	
$p_1 \neq p_2$, $p_1 \neq 1 - p_2$, $p_2 \neq \frac{1}{2}$	$(\pi(h^{\downarrow}), \pi(h^{*\downarrow}))$	$(\pi(h^{\uparrow}), \pi(h^{*\uparrow}))$
	$(\pi(h^{\downarrow}), \pi(h^{*\uparrow}))$	$(\pi(h^{\uparrow}), \pi(h^{*\downarrow}))$
$p_1 \neq p_2$, $p_1 \neq 1 - p_2$, $p_2 = \frac{1}{2}$	$(\pi(h^{\downarrow}), \pi(h^{*\downarrow}))$	$(\pi(h^{\uparrow}), \pi(h^{*\uparrow}))$
	$(\pi(h^{\downarrow}), \pi(h^{*\downarrow}))$	$(\pi(h^{\uparrow}), \pi(h^{*\uparrow}))$

Remarks on the irreducible ⊠-module $(\pi\tau_{ab}, V)$

restrictions	**isomorphic ⊠-modules**		**the space V can be expressed as**
$p_1 \neq p_2$, $p_1 \neq 1 - p_2$, $p_2 \neq \frac{1}{2}$	$(\pi\tau_{\downarrow\downarrow}, V)$	$(\pi\tau_{\uparrow\uparrow}, V)$	$V_d(1-p_1) \otimes V_p(1-p_2)$
	$(\pi\tau_{\downarrow\uparrow}, V)$	$(\pi\tau_{\uparrow\downarrow}, V)$	$V_d(1-p_1) \otimes V_p(p_2)$
$p_1 \neq p_2$, $p_1 \neq 1 - p_2$, $p_2 = \frac{1}{2}$	$(\pi\tau_{\downarrow\downarrow}, V)$	$(\pi\tau_{\uparrow\uparrow}, V)$	$V_d(1-p_1) \otimes V_p\left(\frac{1}{2}\right)$
	$(\pi\tau_{\downarrow\uparrow}, V)$	$(\pi\tau_{\uparrow\downarrow}, V)$	

Summary

In this presentation, we constructed four TD pairs of Krawtchouk type on the finite-dimensional irreducible $\mathfrak{so}_4$-module $V = V(d,p)$. Using these TD pairs, we displayed four Lie algebra homomorphisms $\boxtimes \to \mathfrak{so}_4$. Consequently, we expressed V as a tensor product of two evaluation modules in more than one way.

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THANK YOU 😊