# Almost commutative Terwilliger algebras

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# Origin story

- KG 8? years ago: talk by Paul on subconstituent algebra
- Nick Bastian asks to work on MSc thesis
- NB produces MSc thesis - referenced
- NB continues with work on Terwillger algebra

# Context

Algebra - c.f Allen's talk

# Goal

Goals:

1. To characterize those finite groups that have an almost commutative (AC) Terwilliger algebra.

2. To characterize those strong Gelfand pairs $(G, H), H \leq G$, that have an almost commutative Terwilliger algebra.

3. To define all the terms in the above.

# Association Schemes

An class $d$ association scheme $\mathcal{A}$ on a finite set $X$, $r = |X|$, is a set of nonzero $r \times r$ commuting $0,1$-matrices $A_0, A_1, \ldots, A_d \in M_{|X|}(\mathbb{C})$, where

- $A_0 = I_{|X|}$;
- $\{A_0, A_1, \ldots, A_d\}$ is invariant under transpose
- for $i, j \in \{0, 1, \ldots, d\}$ we have $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$;
- $\sum_{i=0}^{d} A_i = J_r$ is the all 1 matrix.

$A_0, A_1, \cdots, A_d$ are the adjacency matrices - rows and columns are indexed by $X$.

$\mathcal{M} = \mathrm{Span}_{\mathbb{C}}(A_0, A_1, A_2, \cdots, A_d)$ is the Bose-Mesner algebra

Let $E_i$ be the primitive idempotents of $\mathcal{M}$; then $E_i \circ E_j = \sum_{k=0}^{d} q_{ij}^k E_k$.

# $E_i^*(x)$ Matrices

## $E_i^*(x)$ Matrices

Let $\mathcal{A}$ be as above. Let $x \in X$ and define

$$(E_i^*(x))_{y,y} = (A_i)_{x,y}$$

The Terwilliger algebra $\mathcal{T}(x)$ of $\mathcal{A}$ with base point $x$ is the subalgebra generated by all the $A_i$ and $E_i^*(x)$.

## Proposition [Terwilliger]

If $|X| > 1$, then $\mathcal{T}(x)$ is non-commutative and semi-simple.

# Schur rings

For a group $G$ and $S \subseteq G$, we let $\overline{S} = \sum_{x \in S} x \in \mathbb{C}G$ and $S^{-1} = \{x^{-1} : x \in S\}$.

A Schur-*ring* (or *S-ring*) over a group $G$ is a sub-ring $\mathfrak{S}$ of $\mathbb{C}G$ constructed from a partition $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_d\}$ of $G$ with $\Gamma_0 = \{id\}$, satisfying:

(1) invariant under inverse map;
(2) if $0 \leq i, j \leq d$, then

$$\overline{\Gamma}_i \overline{\Gamma}_j = \sum_{k=0}^{d} \lambda_{ijk} \overline{\Gamma}_k,$$

where $\lambda_{ijk} \in \mathbb{Z}^{\geq 0}$ for all $i, j, k$.

The $\Gamma_i$ are called the *principal sets* of the S-ring.

# Group Association Scheme

Let $C_0 = \{1\}, C_1, \ldots, C_d$ be the principal sets for a commutative Schur ring over group $G$. Define

$$(A_i)_{x,y} = \begin{cases} 1 & yx^{-1} \in C_i \\ 0 & \text{otherwise.} \end{cases}$$

This gives an association scheme over $X = G$.

Important example: if the $C_i$ are the conjugacy classes of $G$, then this gives the group association scheme $\mathcal{G}(G)$.

Let $\mathcal{T}(G)$ be the Terwilliger algebra for $\mathcal{G}(G)$, with $x = id(G)$.
The choice of $x \in X$ here is not important.

# Primary ideal

If $X = G = \{g_1, \cdots, g_n\}$ and $R_i = \{g_{i_1}, \cdots, g_{i_r}\}, R_j = \{g_{j_1}, \cdots, g_{j_s}\}$ are principal sets of a commutative Schur ring $\mathfrak{S}$, then the matrices $V_{i,j}$ with entry 1 at every $i_k, j_m$ entry is a basis element for the *primary ideal* $V$ of $\mathcal{T}(G, \mathfrak{S})$. So

$$\dim V = (d+1)^2.$$

# Motivating Table

| Group | Dimension $\mathcal{T}(G)$ | Wedderburn Components |
|---|---|---|
| $S_3$ | 11 | 1, 1, 3 |
| $D_8$ | 28 | 1, 1, 1, 5 |
| $Q_8$ | 28 | 1, 1, 1, 5 |
| $A_4$ | 19 | 1, 1, 1, 4 |
| $F_{20}$ | 29 | 1, 1, 1, 1, 5 |
| $G_{27,3} = 3^{1+2}$ | 137 | 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 11 |
| $G_{27,4} = 3^{1+2}$ | 137 | 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 11 |
| $G_{32,49} = 2^{1+4}$ | 304 | 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 17 |
| $G_{32,50} = 2^{1+4}$ | 304 | 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 17 |

$G$ is *extra special* if $Z(G) = G'$ and $G/G'$ is elementary abelian.

# Question

**Problem of interest:** When does the Wedderburn decomposition of the Terwilliger algebra for a group association scheme consist only of the primary component and $1-$dimensional components?

### Almost Commutative Terwilliger Algebra (Tanaka)

Definition: We say that a Terwilliger algebra $\mathcal{T}(x)$ is almost commutative (AC) if every non-primary irreducible $\mathcal{T}(x)-$module is $1-$dimensional.

## ACTerwilliger Algebra

**Proposition** [Tanaka] Let $\mathcal{A}$ be a commutative association scheme. Let $\mathcal{T}(x)$ be the Terwilliger algebra for $\mathcal{A}$ for some $x \in X$. The following are equivalent:

(1 $\mathcal{T}(x)$ is AC.

(2 Every non-primary irreducible $\mathcal{T}(x)-$module is 1$-$dimensional for *all* $x \in X$.

(3 The $p_{i,j}^k$ satisfy: for distinct $h, i$ there is only one $j$ such that $p_{ij}^h \neq 0$.

(4 The $q_{ij}^k$ satisfy: for distinct $h, i$ there is only one $j$ such that $q_{ij}^h \neq 0$.

(5 $\mathcal{A}$ is the wreath product of association schemes $\mathcal{A}_1, \cdots, \mathcal{A}_w$ where each $\mathcal{A}_i$ is either a trivial (one class) scheme or is the group scheme of a finite abelian group.

# Camina Groups

Recall: A *Frobenius group* is $G \leq S_n$ where each non-trivial element fixes at most one element. In fact $G = N \rtimes H$ for non-trivial subgroups $N, H$. $H$ is called the *Frobenius complement* and $N$ is the *Frobenius kernel*. Say: $H$ acts on $N$ via a *Frobenius action*.

**Camina Group**

A nonabelian group $G$ is called a Camina group if every conjugacy class of $G$ outside of $G'$ is a coset of $G'$.

These are a generalization of Frobenius groups and extra-special groups. Note: All the groups in the above table are Camina groups.

# Camina groups ctd.

**Proposition [Dark and Scoppola]**

Let $G$ be a Camina group. Then one of the following is true:

- $G$ is a Frobenius group whose Frobenius complement is cyclic.
- $G$ is a Frobenius group whose Frobenius complement is $Q_8$.
- $G$ is a $p-$group for some prime $p$ where the nilpotency class of $G$ is either 2 or 3.

# Useful Property

## Theorem

Let $G$ be a group such that $\mathcal{T}(G)$ is AC. Then for all $x, y \in G$ where $x^G \neq (y^{-1})^G$, we have $x^G y^G = (xy)^G$.

## Dade and Yadav

$G$ satisfies the useful property if and only if $G$ is isomorphic to one of:

- An abelian group
- A non-abelian Camina $p-$group.
- A Frobenius group of the form $\mathcal{C}_p^r \rtimes \mathcal{C}_{p^r-1}$.
- The Frobenius group $\mathcal{C}_3^2 \rtimes Q_8$.

# Solution

**Theorem**

Then $\mathcal{T}(G)$ is AC if and only if $G$ is one of

- An abelian group
- A Camina $p-$group (nilpotency class 2 or 3)
- A Frobenius group of the form $\mathcal{C}_p^r \rtimes \mathcal{C}_{p^r-1}$.
- The Frobenius group $\mathcal{C}_3^2 \rtimes Q_8$.

# Dimensions

$G$ a finite abelian group. Then $\dim \mathcal{T}(G) = |G|^2$.

$G = \mathcal{C}_3^2 \rtimes Q_8$. Then $\dim \mathcal{T}(G) = 44$.

$G = \mathcal{C}_p^n \rtimes \mathcal{C}_{p^n-1}$, $n \geq 1$. Then $\dim \mathcal{T}(G) = p^{2n} + p^n - 1$.

$G$ a Camina $p-$group of nilpotency class 2, $|G| = p^n, |Z(G)| = p^k$ Then

$$\dim \mathcal{T}(G) = (p^{n-k} - 1 + p^k)^2 + (p^k - 1)(p^{n-k} - 1).$$

Let $G$ be a Camina $p-$group of nilpotency class 3 where
$|Z(G)| = p^k$, $[G \colon G'] = p^{2n}$ and $[G' \colon Z(G)] = p^n$.
Here $n \in 2\mathbb{Z}$. Then

$$\dim \mathcal{T}(G) = (p^{2n} + p^n + p^k - 2)^2 + (p^n - 1)((p^k - 1)p^n + 2p^k + p^{2n} - 3).$$

# Summary, so far:

We have:

- a classification of all groups $G$ where $\mathcal{T}(G)$ is AC
- found $\dim \mathcal{T}(G)$
- we can also give the idempotents of $\mathcal{T}(G)$
- and a description of the association scheme as a wreath product.

# END OF PART ONE

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# PART TWO: STRONG GELFAND PAIRS

Definition: A *Gelfand pair (GP)* is $(G, H), H \leq G$, where the *double coset algebra* generated by the double cosets $HgH, g \in G$, is commutative

Definition: A *Strong Gelfand pair (SGP)* is $(G, H), H \leq G$, where the *H-class Schur ring* generated by the *H-classes* $\{g^h : h \in H\}$ is commutative

Results of Travis and Karloff show that this definition is equivalent to the character-theoretic and module-theoretical definitions

Example 1. $(G, G)$ is always a SGP

Example 2. $(S_{n+1}, S_n), (S_n, A_n)$ are SGPs

Example 3. $(\mathrm{GL}(n+1, F), \mathrm{GL}(n, F))$ for some fields $F$.

# WREATH PRODUCTS OF ASSOCIATION SCHEMES

Let $\mathcal{C} : C_0, C_1, \cdots, C_r$ and $\mathcal{D} : D_0, D_1, \cdots, D_s$ be association schemes ($C_i$ are $m \times m$ matrices, $D_j$ are $n \times n$ matrices). Then the *wreath product association scheme* is determined by

$$D_0 \otimes C_0 = I_{nm}, \quad I_n \otimes C_i, \ \ i > 0, \quad D_i \otimes J_m, \ \ i > 0.$$

Notation: $\mathcal{C} \wr \mathcal{D}$.

This is an associative product

# Strong Gelfand pair Classification

**Theorem** Let $H \leq G$. Then $(G, H)$ is a Strong Gelfand pair and $\mathcal{T}(G, \mathbb{C}[G]^H)$ is an AC Terwilliger algebra if and only if

1. $H = G$ and $\mathcal{T}(G)$ is AC (see previous result).
2. $G$ is an abelian group with $H < G$. In this case,

$$\dim \mathcal{T}(G, \mathbb{C}[G]^H) = |G|^2.$$

3. $G = H \rtimes \mathcal{C}_k$ is a Frobenius group with Frobenius kernel $H$ and cyclic complement $\mathcal{C}_k$ such that $\mathcal{T}(H)$ is AC . The corresponding association scheme is

$$\mathcal{G}(H) \wr \mathcal{G}(G/H).$$

   If $H$ has $m$ conjugacy classes then

$$\dim \mathcal{T}(G, \mathbb{C}[G]^H) = \dim T(H) + (k-1)(k-2+3m).$$

# Possibilities for Case 3: $G$ is Frobenius

Possibilities for $H$ in Case 3: $G = H \rtimes \mathcal{C}_k$ is Frobenius

As $\mathcal{T}(H)$ is AC, $H$ is one of:

1. an abelian group;
2. a Frobenius group;
3. a Camina $p$-group of nilpotency class 2;
4. a Camina $p$-group of nilpotency class 3.

In each case we have to also find an automorphism of $H$ fixing only the identity of $H$.

So: 2 is not possible by Thompson's theorem - a Frobenius group is not nilpotent.

# Extraspecial groups

Extraspecial groups $H$ are Camina and of class 2.
Two types for $H$: exponent $p$ or $p^2$.
Result of Winter implies any automorphism of extraspecial $H$ of exponent $p^2$ has a non-trivial fixed point in $H$.
Let $H, |H| = p^3$, be extraspecial of exponent $p$:

$$H = \langle a, b, c | a^p, b^p, c^p, c = a^{-1}b^{-1}ab, ac = ca, bc = cb \rangle.$$

Then $\varphi : H \to H$,

$$\varphi(a) = a^k, \quad \varphi(b) = b^k, \quad \varphi(c) = c^{k^2},$$

determines an automorphism that only fixes $1_H$. Here the order of $k$ mod $p$ needs to be an odd prime dividing $p-1$ and $k^2 \not\equiv 1 \mod p$.

# Family of Nilpotent Class 2 Camina $p$-groups that work

Let $p > 2$ be prime. Let $H = \langle h_1, \cdots, h_6 \rangle$ where

$$h_2^{h_1} = h_2 h_4, \quad h_3^{h_1} = h_3 h_5, \quad h_3^{h_2} = h_3 h_6$$

where $h_i^p = 1, 1 \leq i \leq 6$ and $h_4, h_5, h_6$ are central so that $\langle h_4, h_5, h_6 \rangle \cong \mathcal{C}_p^3$. Then $H$ has nilpotency class 2 and is a Camina group of order $p^6$.

Let $X = (x_{ij}) \in \mathrm{SL}(3, p)$. We want $\varphi \in \mathrm{Aut}(H)$ such that

$$g_4 := \varphi(h_4) = h_4^{x_{11}} h_5^{x_{21}} h_6^{x_{31}} \;\; g_5 := \varphi(h_5) = h_4^{x_{12}} h_5^{x_{22}} h_6^{x_{32}}$$
$$g_6 := \varphi(h_6) = h_4^{x_{13}} h_5^{x_{23}} h_6^{x_{33}}.$$

Thus we want to find $a_i, b_i, c_i \in \mathbb{Z}/p\mathbb{Z}$ with

$$g_1 = h_1^{a_1} h_2^{a_2} h_3^{a_3}, \quad g_2 = h_1^{b_1} h_2^{b_2} h_3^{b_3}, \quad g_3 = h_1^{c_1} h_2^{c_2} h_3^{c_3}$$

of order $p$ such that $g_2^{g_1} = g_2 g_4, g_3^{g_1} = g_3 g_5, g_3^{g_2} = g_3 g_6$.

If $h_j^{h_i} = h_j h_k$, then $h_i^{h_j} = h_i h_k^{-1}$.
Thus

$$\begin{aligned} g_2^{g_1} &= (h_1^{b_1} h_2^{b_2} h_3^{b_3})^{h_1^{a_1} h_2^{a_2} h_3^{a_3}} \\ &= (h_1^{b_1})^{h_2^{a_2} h_3^{a_3}} \cdot (h_2^{b_2})^{h_1^{a_1} h_2^{a_2} h_3^{a_3}} \cdot (h_3^{b_3})^{h_1^{a_1} h_2^{a_2} h_3^{a_3}} \\ &= h_1^{b_1} h_4^{a_1 b_2 - a_2 b_1} h_5^{a_1 b_3 - a_3 b_1} h_6^{a_2 b_3 - a_3 b_2} \\ &\qquad \times \cdots \\ &= g_2 g_4 \times \cdots = g_2 h_4^{x_{11}} h_5^{x_{21}} h_6^{x_{31}} \times \cdots \end{aligned}$$

# Family of Nilpotent Class 2 Camina $p$-groups (ctd)

and so we need to solve (over $\mathbb{Z}_p$)

$$\begin{bmatrix} a_1b_2 - a_2b_1 & a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 \\ a_1c_2 - a_2c_1 & a_1c_3 - a_3c_1 & a_2c_3 - a_3c_2 \\ b_1c_2 - b_2c_1 & b_1c_3 - c_1b_3 & b_2c_3 - b_3c_2 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{bmatrix} \qquad (1)$$

Let

$$Y = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Note: LHS of (1) is $\Lambda^2(Y)$.

# Family of Nilpotent Class 2 Camina $p$-groups (ctd)

Let $A = (a_1, a_2, a_3), B = (b_1, b_2, b_3), C = (c_1, c_2, c_3)$. Now letting $\times$ denote the standard cross product on 3-vectors we have:

$$A \times B = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)^T = (x_{31}, -x_{21}, x_{11})^T;$$
$$B \times C = (b_2 c_3 - b_3 c_2, b_3 c_1 - b_1 c_3, b_1 c_2 - b_2 c_1)^T = (x_{33}, -x_{23}, x_{13})^T;$$
$$C \times A = (c_2 a_3 - c_3 a_2, c_3 a_1 - c_1 a_3, c_1 a_2 - c_2 a_1)^T = (-x_{32}, x_{22}, -x_{12})^T.$$

A standard identity is:

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B).$$

This gives

$$x_{33} a_1 - x_{23} a_2 + x_{13} a_3 = -x_{32} b_1 + x_{22} b_2 - x_{12} b_3 \tag{2}$$
$$= x_{31} c_1 - x_{21} c_2 + x_{11} c_3. \tag{3}$$

# Family of Nilpotent Class 2 Camina $p$-groups (ctd)

We also have the identities:

$$A \cdot (A \times B) = B \cdot (A \times B) = A \cdot (A \times C) = 0;$$
$$C \cdot (A \times C) = B \cdot (B \times C) = C \cdot (B \times C) = 0.$$

These latter give:

$$x_{31}a_1 - x_{21}a_2 + x_{11}a_3 = 0; \tag{4}$$
$$x_{31}b_1 - x_{21}b_2 + x_{11}b_3 = 0; \tag{5}$$
$$x_{32}a_1 - x_{22}a_2 + x_{12}a_3 = 0; \tag{6}$$
$$x_{32}c_1 - x_{22}c_2 + x_{12}c_3 = 0; \tag{7}$$
$$x_{33}b_1 - x_{23}b_2 + x_{13}b_3 = 0; \tag{8}$$
$$x_{33}c_1 - x_{23}c_2 + x_{13}c_3 = 0. \tag{9}$$

# Family of Nilpotent Class 2 Camina $p$-groups (ctd)

Solving this system of eight linear equations Eqs (2)-(9) for the eight variables $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2$ gives

$$\begin{aligned}
a_1 &= -(-x_{11}x_{22} + x_{12}x_{21})/(x_{22}x_{33} - x_{23}x_{32})c_3, \qquad (10)\\
a_2 &= (x_{11}x_{32} - x_{12}x_{31})/(x_{22}x_{33} - x_{23}x_{32})c_3,\\
a_3 &= (x_{21}x_{32} - x_{22}x_{31})/(x_{22}x_{33} - x_{23}x_{32})c_3,\\
b_1 &= (x_{11}x_{23} - x_{13}x_{21})/(x_{22}x_{33} - x_{23}x_{32})c_3,\\
b_2 &= (x_{11}x_{33} - x_{13}x_{31})/(x_{22}x_{33} - x_{23}x_{32})c_3,\\
b_3 &= (x_{21}x_{33} - x_{23}x_{31})/(x_{22}x_{33} - x_{23}x_{32})c_3,\\
c_1 &= (x_{12}x_{23} - x_{13}x_{22})/(x_{22}x_{33} - x_{23}x_{32})c_3,\\
c_2 &= (x_{12}x_{33} - x_{13}x_{32})/(x_{22}x_{33} - x_{23}x_{32})c_3.
\end{aligned}$$

One then finds that $c_3$ satisfies

$$c_3^2 = \frac{(x_{22}x_{33} - x_{23}x_{32})^2}{\det X} = (x_{22}x_{33} - x_{23}x_{32})^2,$$

# Family of Nilpotent Class 2 Camina $p$-groups (ctd)

Taking $c_3 = x_{22}x_{33} - x_{23}x_{32}$ we get

$$\begin{aligned} a_1 &= x_{11}x_{22} - x_{12}x_{21}, \\ a_2 &= x_{11}x_{32} - x_{12}x_{31}, \\ a_3 &= x_{21}x_{32} - x_{22}x_{31}, \\ b_1 &= x_{11}x_{23} - x_{13}x_{21}, \qquad \textit{etc}. \end{aligned} \tag{11}$$

POINT: If we take $X \in \mathrm{SL}(3,p)$ with eigenvalues not equal to 1 and of prime order $q \neq p$, then the above gives $Y$ solving the equations.

Then $X$ and $Y$ determine $\varphi \in \mathrm{Aut}(H)$ that gives a Frobenius action of $\langle\varphi\rangle \cong \mathcal{C}_q$ on $H$ and so a Frobenius group

$$G = H \rtimes_\phi \mathcal{C}_q$$

which gives a strong Gelfand pair $(G, H)$ such that $\mathcal{T}(G, \mathbb{C}G^H)$ is AC .

# Example of a Nilpotent Class 3 Camina $p$-groups with a Frobenius automorphism

Take $H = G_{11^7,750208}$ - a Camina 11-group of class 3.

Then there is a fixed-point-free automorphism $\varphi$ of $H$ of order 5 which gives a Frobenius action and so a Frobenius group

$$G = H \rtimes_{\varphi} \mathcal{C}_5$$

with AC Terwilliger algebra - found using Magma.

# THE END

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