A correspondence between Leonard pairs, orthogonal polynomials and the algebraic Bethe ansatz

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→ **Paul Terwilliger's Fest** ←

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In the context of quantum algebra, representation theory and combinatorics, from the 90's the **Askey-Wilson algebra AW** [Zhedanov,'91] and the **theory of Leonard pairs** [Terwilliger et al.,'99] have been introduced and studied.

Theory of Leonard pairs (LP) gives :

→ a classification of finite dim. irreps. of **AW algebra** over $\mathbb{K}$ (q not root of 1) [Terwilliger-Vidunas,'03]

$$\pi : AW \to \mathrm{End}(\mathcal{V}) \qquad \pi(\mathsf{A}) = A, \quad \pi(\mathsf{A}^*) = A^*$$

→ an interpretation of **discrete orthogonal polynomials** of the Askey-scheme
3-term recurrence relation, difference equation, Askey-Wilson duality, orthogonality satisfied by $R_M(\theta^*_N)$ [Zhedanov,'91],[Terwilliger,'03]

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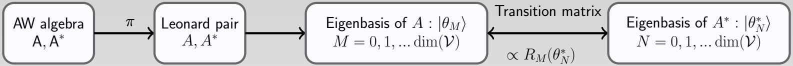

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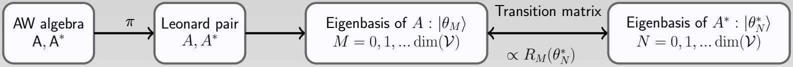

Other material needed :

$A^\diamond = a[A^*, A]_q + b$
$\hookrightarrow$ Leonard triple $A, A^*, A^\diamond$

Dual eigenbasis of A (or A^*) : $\langle\theta_M|$ (or $\langle\theta^*_N|$)
$M, N = 0, 1, \dots \dim(\tilde{\mathcal{V}})$

Notation : $[X, Y]_q = qXY - q^{-1}YX$.

In the context of mathematical physics, from the 70-80's Leningrad school developped the **algebraic Bethe ansatz** [Faddeev et al.].

Algebraic Bethe ansatz (ABA) gives :

→ **explicit eigenvectors** of **arbitrary** combinations of generating functions

$$\mathscr{A}^{\epsilon}(u,m), \mathscr{B}^{\epsilon}(u,m), \mathscr{C}^{\epsilon}(u,m), \mathscr{D}^{\epsilon}(u,m) \in \mathcal{A} \otimes \mathbb{C}[u,u^{-1}]$$

satisfying exchange relations inherited from the **Yang-Baxter algebras YB** [Yang,'68 ; Baxter,'71] or **reflection equation algebras RE** [Cherednik '82 ; Sklyanin,'88] associated with the 'FRT' presentation for algebra $\mathcal{A}$

→ **scalar products of eigenvectors** as **determinant** of some matrices [Gaudin '72 ; Korepin,'82]

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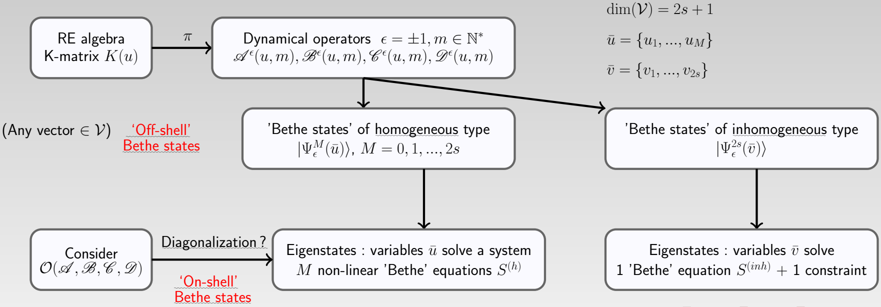
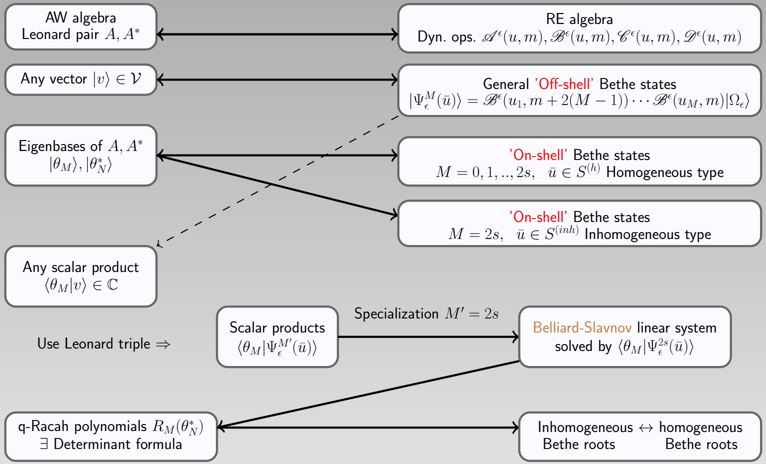


Correspondence Leonard pairs/triples of q-Racah type $\longleftrightarrow$ algebraic Bethe ansatz



Toy model ! $\rightarrow$ Towards **Correspondence** between the **theory of tridiagonal pairs** (q-Onsager algebra finite dim. irreps. [Ito-Terwiliger,'09]) $\longleftrightarrow$ **Algebraic Bethe ansatz**

AW algebra (Leonard pair A, A^*)	⟷ **Step 1 (& 2)**	RE algebra (Dyn. ops. $\mathscr{A}^\epsilon(u,m), \mathscr{B}^\epsilon(u,m), \mathscr{C}^\epsilon(u,m), \mathscr{D}^\epsilon(u,m)$)

- The **Askey-Wilson algebra** (over $\mathbb{C}$) : generators A, A* [Zhedanov,'91]

$$[\mathsf{A}, [\mathsf{A}, \mathsf{A}^*]_q]_{q^{-1}} = \rho\, \mathsf{A}^* + \omega\, \mathsf{A} + \eta \mathcal{I}\ , \qquad q, \rho, \omega, \eta, \eta^* \in \mathbb{C}$$

$$[\mathsf{A}^*, [\mathsf{A}^*, \mathsf{A}]_q]_{q^{-1}} = \rho\, \mathsf{A} + \omega\, \mathsf{A}^* + \eta^* \mathcal{I} \qquad \text{with} \qquad [X, Y]_q = qXY - q^{-1}YX.$$

- The **reflection equation algebra presentation of AW** [B,'04] :

$$R(u/v)\ (\mathcal{K}(u) \otimes I\!I)\ R(uv)\ (I\!I \otimes \mathcal{K}(v))\ =\ (I\!I \otimes \mathcal{K}(v))\ R(uv)\ (\mathcal{K}(u) \otimes I\!I)\ R(u/v)\ .$$

→ u, v indeterminates,

→ $\mathcal{K}(u) \in AW[u, u^{-1}] \otimes \mathrm{End}(\mathbb{C}^2)$ called **'K-operator'**

→ $R(u) \in \mathbb{C}[u, u^{-1}] \otimes \mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^2)$ called **'R-matrix'**

Correspondence : AW algebra $\longleftrightarrow$ Reflection equation algebra

$\rightarrow$ Starting point for ABA construction

The **R-matrix** : $$R(u)=\begin{pmatrix} uq-u^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & u-u^{-1} & q-q^{-1} & 0 \\ 0 & q-q^{-1} & u-u^{-1} & 0 \\ 0 & 0 & 0 & uq-u^{-1}q^{-1} \end{pmatrix}$$

and the **K-operator** : $$\mathcal{K}(u)=\begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} ,$$

$$\mathcal{A}(u)=(u^2-u^{-2})\left(qu\,\mathsf{A}-q^{-1}u^{-1}\mathsf{A}^*\right)-(q+q^{-1})\rho^{-1}\left(\eta u+\eta^* u^{-1}\right) ,$$
$$\mathcal{D}(u)=(u^2-u^{-2})\left(qu\,\mathsf{A}^*-q^{-1}u^{-1}\mathsf{A}\right)-(q+q^{-1})\rho^{-1}\left(\eta^* u+\eta u^{-1}\right) ,$$
$$\mathcal{B}(u)=\chi(u^2-u^{-2})\left(\rho^{-1}\left([\mathsf{A}^*,\mathsf{A}]_q+\frac{\omega}{q-q^{-1}}\right)+\frac{qu^2+q^{-1}u^{-2}}{q^2-q^{-2}}\right) , \quad (\chi\in\mathbb{C}^*)$$
$$\mathcal{C}(u)=\rho\chi^{-1}(u^2-u^{-2})\left(\rho^{-1}\left([\mathsf{A},\mathsf{A}^*]_q+\frac{\omega}{q-q^{-1}}\right)+\frac{qu^2+q^{-1}u^{-2}}{q^2-q^{-2}}\right)$$

solve $$R(u/v)\,(\mathcal{K}(u)\otimes \mathbb{I})\,R(uv)\,(\mathbb{I}\otimes\mathcal{K}(v)) \;=\; (\mathbb{I}\otimes\mathcal{K}(v))\,R(uv)\,(\mathcal{K}(u)\otimes\mathbb{I})\,R(u/v) .$$

(AW algebra) Leonard pair A, A^* ⟷ **(Step 1 completed) & 2** ⟷ (RE algebra) Dyn. ops. $\mathscr{A}^\epsilon(u,m), \mathscr{B}^\epsilon(u,m), \mathscr{C}^\epsilon(u,m), \mathscr{D}^\epsilon(u,m)$

Definition : A **Leonard pair** on $\mathcal{V}$ is an ordered pair of diagonalizable linear maps $A : \mathcal{V} \to \mathcal{V}$ and $A^* : \mathcal{V} \to \mathcal{V}$ that each act on an eigenbasis for the other one in an irreducible tridiagonal fashion.

Recall **AW algebra** with generators A, A^*...

$$\big[\mathsf{A}, [\mathsf{A}, \mathsf{A}^*]_q\big]_{q^{-1}} = \rho\,\mathsf{A}^* + \omega\,\mathsf{A} + \eta\mathcal{I}\ , \qquad q, \rho, \omega, \eta, \eta^* \in \mathbb{C}$$
$$\big[\mathsf{A}^*, [\mathsf{A}^*, \mathsf{A}]_q\big]_{q^{-1}} = \rho\,\mathsf{A} + \omega\,\mathsf{A}^* + \eta^*\mathcal{I} \qquad \text{with} \qquad [X, Y]_q = qXY - q^{-1}YX.$$

→ **Theorem** [Terwilliger-Vidunas,'03]

Let $(\pi, \mathcal{V})$ be a finite dim. rep. such that

- $\pi(\mathsf{A}), \pi(\mathsf{A}^*)$ are diagonalizable and spectra non-degenerate ;
- $\mathcal{V}$ is irreducible.

Then $\pi(\mathsf{A}), \pi(\mathsf{A}^*)$ is called a **Leonard pair**.

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Consider a **Leonard pair A, A^* of q-Racah type** $\pi : AW \to \mathrm{End}(\mathcal{V})$, $\quad \dim(\mathcal{V}) = 2s+1$

$$\mathsf{A} \mapsto A \ , \quad \mathsf{A}^* \mapsto A^*$$

Spectra : $\theta_M = \mathsf{b}q^{2M} + \mathsf{c}q^{-2M}$, $\theta^*_N = \mathsf{b}^*q^{2N} + \mathsf{c}^*q^{-2N}$, $\quad (M, N = 0, 1, ...2s)$.

Tridiagonal structure : $\exists$ **eigenbasis** of A (resp. A^*) $|\theta_M\rangle$ (resp. $|\theta^*_N\rangle$) such that

$$\begin{aligned} A|\theta_M\rangle &= \theta_M|\theta_M\rangle \ , \quad A^*|\theta_M\rangle = A^{(*,.)}_{M+1,M}|\theta_{M+1}\rangle + A^{(*,.)}_{M,M}|\theta_M\rangle + A^{(*,.)}_{M-1,M}|\theta_{M-1}\rangle \ , \\ A^*|\theta^*_N\rangle &= \theta^*_N|\theta^*_N\rangle \ , \quad A|\theta^*_N\rangle = A^{(.,*)}_{N+1,N}|\theta^*_{N+1}\rangle + A^{(.,*)}_{N,N}|\theta^*_N\rangle + A^{(.,*)}_{N-1,N}|\theta^*_{N-1}\rangle \ . \end{aligned}$$

$(A^{(*,.)}_{-1,0} = A^{(*,.)}_{2s+1,2s} = A^{(.,*)}_{-1,0} = A^{(.,*)}_{2s+1,2s} = 0)$ [Terwilliger,'03]

→ In the **ABA setting** for Leonard pairs, we need the image via the algebra map $\pi : AW \to \mathrm{End}(\mathbb{C}^{2s+1})$ of the K-operator $\mathcal{K}(u) \in AW[u,u^{-1}] \otimes \mathrm{End}(\mathbb{C}^2)$:

$$(\pi \otimes \mathrm{id})\mathcal{K}(u) = \begin{pmatrix} \pi(\mathcal{A}(u)) & \pi(\mathcal{B}(u)) \\ \pi(\mathcal{C}(u)) & \pi(\mathcal{D}(u)) \end{pmatrix} ,$$

Remark : $\pi(\mathcal{A}(u))$,... are combinations of matrices of size $2s+1 \times 2s+1$. Entries $\pi(\mathcal{A}(u))_{ij}, ... \in \mathbb{C}[u,u^{-1}]$.

For instance : $\pi(\mathcal{C}(u)) = \rho\chi^{-1}\left(u^2 - u^{-2}\right)\left(\rho^{-1}\left([A,A^*]_q + \frac{\omega}{q-q^{-1}}\right) + \frac{qu^2+q^{-1}u^{-2}}{q^2-q^{-2}}\right)$.

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→ In the **ABA setting** associated with the reflection equation algebra, certain polynomials of $\pi(\mathcal{A}(u)), \pi(\mathcal{B}(u)), \pi(\mathcal{C}(u)), \pi(\mathcal{D}(u))$ called '**dynamical operators**' are required for constructing Bethe states. Depend on extra scalars α, β [Cao et al. '03] :

Dynamical operators : $\{\mathscr{A}^\epsilon(u,m), \mathscr{B}^\epsilon(u,m), \mathscr{C}^\epsilon(u,m), \mathscr{D}^\epsilon(u,m) | m \in \mathbb{N}^*, \epsilon = \pm 1\}$

Example for the RE associated with a Leonard pair A, A^* :

$$\mathscr{B}^+(u,m) = \frac{\beta(u^2-u^{-2})}{\alpha q^{2m+2}-\beta}\Big(\frac{\chi q^m}{\beta\rho u}[A^*,A]_q - \frac{\beta q^{-m}}{u\chi}[A,A^*]_q + \frac{(q^2+1)}{qu}A - (\frac{1}{qu^3}+qu)A^*$$
$$+f(u,u^{-1};m,\beta,\rho,\omega,\eta,\eta^*,\chi)$$

Correspondence : $A, A^* \longleftrightarrow \mathscr{A}^\epsilon(u,m), \mathscr{B}^\epsilon(u,m), \mathscr{C}^\epsilon(u,m), \mathscr{D}^\epsilon(u,m)$

$\rightarrow$ Ready for ABA construction !

$\rightarrow$ Case $\epsilon = -$

$$A = \frac{u^{-1}}{b(u^2)}\left(\frac{1}{b(qu^2)}\mathscr{A}^-(u,m) + \frac{1}{b(q^2u^2)}\mathscr{D}^-(u,m)\right) + \frac{q\,u\,\bar{\eta}(u) + q^{-1}u^{-1}\bar{\eta}(u^{-1})}{b(u^2)(b(q^2u^2)} \,,$$

$$A^* = \text{linear combination of } \mathscr{A}^-(u,m), \mathscr{B}^-(u,m), \mathscr{C}^-(u,m), \mathscr{D}^-(u,m) \,.$$

$\rightarrow$ Case $\epsilon = +$

$$A = \text{linear combination of } \mathscr{A}^+(u,m), \mathscr{B}^+(u,m), \mathscr{C}^+(u,m), \mathscr{D}^+(u,m) \,,$$

$$A^* = \frac{u}{b(u^2)}\left(\frac{1}{b(qu^2)}\mathscr{A}^+(u,m) + \frac{1}{b(q^2u^2)}\mathscr{D}^+(u,m)\right) + \frac{q\,u\,\bar{\eta}(u^{-1}) + q^{-1}u^{-1}\bar{\eta}(u)}{b(u^2)b(q^2u^2)} \,,$$

where $\bar{\eta}(u) = (q+q^{-1})\rho^{-1}\left(\eta u + \eta^* u^{-1}\right), \qquad b(x) = x - x^{-1}.$

Remark : Dynamical operators depend on free scalars α, β and integer m.

Step 3 : Definition of Bethe states for LP

AW algebra
Leonard pair A, A^*

Step 1 & 2 (completed) $\longleftrightarrow$

RE algebra
Dyn. ops. $\mathscr{A}^\epsilon(u,m), \mathscr{B}^\epsilon(u,m), \mathscr{C}^\epsilon(u,m), \mathscr{D}^\epsilon(u,m)$

Any vector $|v\rangle \in \mathcal{V}$

Step 3 $\longleftrightarrow$

General 'Off-shell' Bethe states
$|\Psi_\epsilon^M(\bar{u})\rangle = \mathscr{B}^\epsilon(u_1, m+2(M-1))\cdots\mathscr{B}^\epsilon(u_M,m)|\Omega_\epsilon\rangle$

$\rightarrow$ In **ABA setting**, we need to identify so-called '**reference states**' $|\Omega^\epsilon\rangle$, $\epsilon = \pm$

Recall that the parameters α, β, m enter in the definition of the **dynamical operators**

$$\mathscr{A}^\epsilon(u,m), \mathscr{B}^\epsilon(u,m), \mathscr{C}^\epsilon(u,m), \mathscr{D}^\epsilon(u,m) \qquad m \in \mathbb{N}^*,\ \epsilon = \pm$$

Step 3 : Definition of Bethe states for LP

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Any vector $\|v\rangle \in \mathcal{V}$	**Step 3** ⟷	General 'Off-shell' Bethe states $\|\Psi^M_\epsilon(\bar{u})\rangle = \mathscr{B}^\epsilon(u_1, m+2(M-1)) \cdots \mathscr{B}^\epsilon(u_M, m)\|\Omega_\epsilon\rangle$

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Correspondence : $|\theta_0\rangle \longleftrightarrow |\Omega^-\rangle$, $|\theta_0^*\rangle \longleftrightarrow |\Omega^+\rangle$

Lemmas : If the parameter α is chosen such that :

$$(q^2 - q^{-2})\chi^{-1}\alpha \mathsf{c}^* q^{m_0} = 1 \qquad (\text{resp. } (q^2 - q^{-2})\chi^{-1}\alpha \mathsf{b} q^{-m_0} = -1)$$

then

$$\mathscr{C}^+(u,m_0)|\Omega^+\rangle = 0 \qquad (\text{resp. } \mathscr{C}^-(u,m_0)|\Omega^-\rangle = 0)$$

$$\mathscr{A}^\pm(u,m_0)|\Omega^\pm\rangle = \Lambda_1^\pm(u)|\Omega^\pm\rangle \qquad \text{and} \qquad \mathscr{D}^\pm(u,m_0)|\Omega^\pm\rangle = \Lambda_2^\pm(u)|\Omega^\pm\rangle$$

AW algebra
Leonard pair A, A^*

Step 1 & 2 (completed) $\longleftrightarrow$

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$\rightarrow$ In **ABA setting**, from any of the **reference states** $|\Omega^\epsilon\rangle$and using successive actions of dynamical operators $\mathscr{B}^+(u,m_0)$ we get $\boxed{\text{any vector in } \mathcal{V}}$:

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Leonard pair A, A^*

Step 1 & 2 (completed)

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Consider the string of dynamical operators ($\bar{u} = \{u_1, u_2, \ldots, u_M\}$)

$$B^\epsilon(\bar{u}, m, M) \quad = \quad \mathscr{B}^\epsilon(u_1, m+2(M-1))\cdots\mathscr{B}^\epsilon(u_M, m)$$

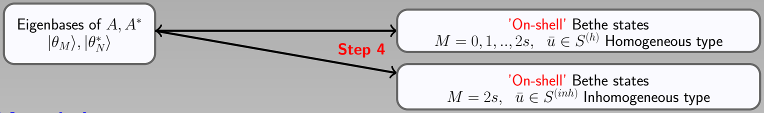
Definitions : The **Bethe states** generated from the Leonard pair A, A^* are :

$$|\Psi_-^M(\bar{u}, m_0)\rangle = B^-(\bar{u}, m_0, M)|\Omega^-\rangle \quad \text{for} \quad (q^2-q^{-2})\chi^{-1}\alpha \mathsf{b} q^{-m_0} = -1 \quad \text{and} \quad \beta = 0\ ,$$

$$|\Psi_+^M(\bar{u}, m_0)\rangle = B^+(\bar{u}, m_0, M)|\Omega^+\rangle \quad \text{for} \quad (q^2-q^{-2})\chi^{-1}\alpha \mathsf{c}^* q^{m_0} = 1 \quad \text{and} \quad \beta = 0\ .$$

Physics's terminology : '**Off-shell**' Bethe states ('**On-shell**' Bethe states if $\bar{u}$ solve BE)

Step 4 : Eigenbases for LP & On-shell Bethe states



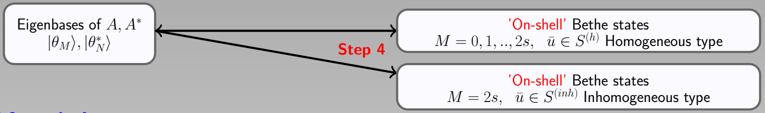
ABA technic :

Let $\mathcal{O}(u) \equiv \mathcal{O}(\mathcal{A}(u), \mathcal{B}(u), \mathcal{C}(u), \mathcal{D}(u))$, where $\mathcal{A}(u), \ldots$ satisfy exchange relations from RE algebra. **Diagonalization of** $\mathcal{O}$?

$$\mathcal{O}(u)|\Psi(\bar{u})\rangle = \lambda(u;\bar{u})|\Psi(\bar{u})\rangle + \sum_{j=1}^{M} E_j(\bar{u})|\Psi_j(\bar{u})\rangle$$

→ $|\Psi(\bar{u})\rangle$ eigenvector of $\mathcal{O}(u)$ if $\bar{u}$ solve 'Bethe Equations' $E_j(\bar{u}) = 0, \ j = 1, ..., M$.

→ Eigenvalue of $\mathcal{O}(u)$ is $\lambda(u;\bar{u})$.



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$\rightarrow$ $|\Psi(\bar{u})\rangle$ eigenvector of $\mathcal{O}(u)$ if $\bar{u}$ solve 'Bethe Equations' $E_j(\bar{u}) = 0, \; j = 1, ..., M$.

$\rightarrow$ Eigenvalue of $\mathcal{O}(u)$ is $\lambda(u;\bar{u})$.

Application to LP A, A^* : Two possibilities

$\bar{u} = \{u_1, \ldots, u_M\}$ solve homogeneous BE
$\bar{v} = \{v_1, \ldots, v_{2s}\}$ solve inhomogeneous BE

'Homogeneous' correspondence :

$$|\theta_M\rangle \longleftrightarrow |\Psi^M_-(\bar{u}, m_0)\rangle \;, \quad |\theta^*_N\rangle \longleftrightarrow |\Psi^N_+(\bar{u}, m_0)\rangle$$

'Inhomogeneous' correspondence :

$$|\theta_M\rangle \longleftrightarrow |\Psi^{2s}_+(\bar{v}, m_0)\rangle \;, \quad |\theta^*_N\rangle \longleftrightarrow |\Psi^{2s}_-(\bar{v}, m_0)\rangle$$

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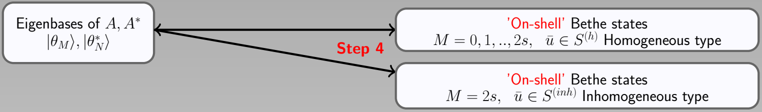

$\rightarrow$ **Hom. Bethe equations** **(system of M coupled equations in M variables)**

$$E_-^M(u_i, \bar{u}_i) = 0 \quad \text{for} \quad \bar{u} = S_-^{M(h)} , \qquad (\text{resp.} \quad E_+^N(u_i, \bar{u}_i) = 0 \quad \text{for} \quad \bar{u} = S_+^{*N(h)})$$

with $$E_\pm^M(u_i, \bar{u}_i) = -\frac{b(u_i^2)}{b(qu_i^2)} \prod_{j=1, j\neq i}^{M} f(u_i, u_j)\Lambda_1^\pm(u_i) + \prod_{j=1, j\neq i}^{M} h(u_i, u_j)\Lambda_2^\pm(u_i) .$$

- $\bar{u} = \{u_1, ..., u_M\}$ and $\bar{u}_i = \bar{u} \backslash u_i$
- $b(x) = x - x^{-1}$
- $\Lambda_{1,2}^\pm(u)$: **rational functions** of u and LP data (scalars $\mathrm{b}, \mathrm{b}^*, \mathrm{c}, \mathrm{c}^*$ in LP spectra)
- $f(u, v), h(u, v)$ data from **exchange relations** between dynamical operators

$\rightarrow$ **Inhom. Bethe equation + 1 constraint (2 equations in $2s$ variables)**

$$E_+(v_j,\bar{v}_j)=0 \quad \text{for} \quad \bar{v}=S_+^{M(i)} \qquad (\text{resp.} \quad E_-(v_j,\bar{v}_j)=0 \quad \text{for} \quad \bar{v}=S_-^{*N(i)})$$

$$\theta_M \;=\; q^{-4s}\Big(\mathsf{c}^*(\zeta^2+\zeta^{-2})[2s]_q+q^{2s}(\mathsf{b}q^{2s}+\mathsf{c}q^{-2s})-q\mathsf{c}^*\sum_{j=1}^{2s}(qu_j^2+q^{-1}u_j^{-2})\Big) \quad \text{for} \quad \bar{u}=S_+^{M(i)}$$

$$(\text{resp.} \;\; \theta_N^* \;=\; q^{4s}\Big(\mathsf{b}(\zeta^2+\zeta^{-2})[2s]_q+q^{-2s}(\mathsf{b}^*q^{2s}+\mathsf{c}^*q^{-2s})-q^{-1}\mathsf{b}\sum_{j=1}^{2s}(qw_j^2+q^{-1}w_j^{-2})\Big) \quad \text{for} \quad \bar{w}=S_-^{*N(i)}\;)$$

with
$$E_\pm(v_i,\bar{v}_i) \;=\; \frac{b(v_i^2)}{b(qv_i^2)}v_i^{\pm1}\prod_{j=1,j\neq i}^{2s} f(v_i,v_j)\Lambda_1^\pm(v_i)-(q^2v_i^3)^{\mp1}\prod_{j=1,j\neq i}^{2s} h(v_i,v_j)\Lambda_2^\pm(v_i)$$
$$+\nu_\pm\frac{v_i^{\mp2}b(v_i^2)}{b(q)}\frac{\prod_{k=0}^{2s}b(q^{1/2+k-s}\zeta v_i)b(q^{1/2+k-s}\zeta^{-1}v_i)}{\prod_{j=1,j\neq i}^{2s}b(v_iv_j^{-1})b(qv_iv_j)}\;,$$

where $\nu_+=q^{-1-4s}\mathsf{c}^*\,,\quad \nu_-=q^{1+4s}\mathsf{b}\,,\quad \zeta\in\mathbb{C}^*.$

- Notation : $\bar{v}=\{v_1,...,v_{2s}\}$ and $\bar{v}_i=\bar{v}\backslash v_i$.

→ **Bethe states** generated from the Leonard pair (A, A^*) are :

$$|\Psi_\epsilon^M(\bar{u}, m_0)\rangle = \mathscr{B}^\epsilon(u_1, m+2(M-1)) \cdots \underbrace{\mathscr{B}^\epsilon(u_M, m)}_{\text{Dynamical op.}} |\Omega^\epsilon\rangle$$

$$\boxed{\mathscr{B}^\epsilon(u,m)) = b(u^2)u^{-\epsilon}\frac{q^{-\epsilon(m+2)}}{\alpha}\left(A^\diamond - \frac{(q+q^{-1})}{r_0}U\right)}$$

- 'Symmetric' variable $U = \frac{qu^2+q^{-1}u^{-2}}{q+q^{-1}}$.
- Scalar r_0 such that $r_0^{-2} = \mathsf{bc} = \mathsf{b}^*\mathsf{c}^* = \mathsf{b}^\diamond\mathsf{c}^\diamond$.
- Dynamical operator in terms of :

$$\boxed{A^\diamond = \frac{r_0}{(q^2-q^{-2})}\left[A^*, A\right]_q + \frac{r_0\ \omega^{\{\cdot,*,\diamond\}}}{(q-q^{-1})(q^2-q^{-2})}\mathbb{I}}$$

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Leonard triple $A, A^*, A^\diamond$ [Curtin,'07 ;Huang,'11] : AW relations '$\mathbb{Z}_3$' form :

$$\frac{r_0}{(q^2-q^{-2})}[A^b, A^a]_q - A^c + \frac{r_0 \ \omega^{\{a,b,c\}}}{(q-q^{-1})(q^2-q^{-2})} = 0$$

⇒ Compute Leonard triple's respective actions on eigenbases of $A, A^*, A^\diamond$ to **decompose Bethe states** into A's eigenvectors.

Proposition : **Bethe states' decompositions** $(M''=0,1,...,2s)$

$$|\Psi_-^{M''}(\bar{u},m)\rangle = \mathcal{G}^-_{M''}(\bar{u}) \sum_{N,M'=0}^{2s} \frac{f_N}{f_0}(P^{\{\diamond,.\}-1})_{NM'} \prod_{i=1}^{M''}\left(U_i - \frac{r_0}{q+q^{-1}}\theta^\diamond_{M'}\right)|\theta_N\rangle$$

$$|\Psi_+^{M''}(\bar{u},m)\rangle = \mathcal{G}^+_{M''}(\bar{u}) \sum_{N,M'=0}^{2s} \frac{f_N\,\nu_0^{\{*,\diamond,.\}-1}}{g_0 h_{M'}}(P^{\{\diamond,.\}-1})_{NM'} \prod_{i=1}^{M''}\left(U_i - \frac{r_0}{q+q^{-1}}\theta^\diamond_{M'}\right)|\theta_N\rangle$$

for certain conditions on spectra data $A, A^*, A^\diamond$ (irreducibility) :

(i) $\mathsf{b}^a(\mathsf{c}^a)^{-1} \neq q^{-2M}$, $\mathsf{b}^b(\mathsf{c}^b)^{-1} \neq q^{-2M}$, $\mathsf{b}^c(\mathsf{c}^c)^{-1} \neq q^{-2M}$, for $1 \leq M \leq 4s-1$,

(ii) $\mathsf{c}^b \neq -r_0 q^{-2M+1}\mathsf{c}^a\mathsf{c}^c$, $\mathsf{b}^a \neq -r_0 q^{-2M+1+4s}\mathsf{b}^b\mathsf{b}^c$, $\mathsf{c}^a \neq -r_0 q^{2M+4s-1}\mathsf{b}^b\mathsf{b}^c$, $\mathsf{c}^b \neq -r_0 q^{2M-1}\mathsf{b}^a\mathsf{c}^c$.

Data :

- $\mathcal{G}^\epsilon_M(\bar{u}) = \left(\epsilon(q+q^{-1})q^{-\epsilon(M+1)}\mathsf{b}^{\frac{1-\epsilon}{2}}\mathsf{c}^{*\,\frac{1+\epsilon}{2}}\right)^M \left(\prod_{i=1}^M b(u_i^2)u_i^{-\epsilon}\right)$, $\quad U_i = \frac{qu_i^2+q^{-1}u_i^{-2}}{q+q^{-1}}$
- $f_M = f_0\prod_{k=0}^{M-1}\frac{\mathsf{c}^a q^{-2k-2}+r_0q^{-1}\mathsf{c}^b\mathsf{b}^c}{\mathsf{b}^a q^{2k}+r_0q^{-1}\mathsf{c}^b\mathsf{b}^c}$, similar type of expressions for g_M, h_M,
- $\frac{f_0}{g_0h_0}$ fixed in terms of ratios of q-Pochammer and $A, A^*, A^\diamond$ spectra data.
- $(P^{\{a,b\}-1})_{NM} = \nu_0^{\{a,b,c\}-1} k_M^{\{b,a,c\}} R_M^{\{a,c\}}(\theta_N^b)$
- **q-Racah pol.** $R_M^{\{a,c\}}(\theta_N^b) = {}_4\phi_3\left[\begin{matrix} q^{-2M},\ \frac{\mathsf{b}^a}{\mathsf{c}^a}q^{2M},\ q^{-2N},\ \frac{\mathsf{b}^b}{\mathsf{c}^b}q^{2N} \\ -\frac{\mathsf{b}^a\mathsf{c}^c}{\mathsf{c}^b}r_0q,\ -\frac{\mathsf{b}^b\mathsf{b}^c}{\mathsf{c}^a}r_0q^{4s+1},\ q^{-4s}\end{matrix};q^2,q^2\right]$

Step 6 : Computing scalar products using Leonard triple

Use properties of Leonard triple $A, A^*, A^\diamond \Rightarrow$ Scalar products $\langle\theta_M|\Psi_\epsilon^{M'}(\bar{u})\rangle$ (Dual basis $\langle\theta_M|$ linear maps $\mathcal{V} \to \mathbb{C}$)

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Theorem : scalar products for arbitrary 'off-shell' Bethe states

$$\frac{\langle\theta_M|\Psi_-^{M''}(\bar{u},m)\rangle}{\langle\theta_M|\theta_M\rangle} = \mathcal{G}^-_{M''}(\bar{u})\frac{f_M}{f_0}\sum_{M'=0}^{2s}(P^{\{\diamond,.\}^{-1}})_{MM'}\prod_{i=1}^{M''}\left(U_i-\frac{r_0}{q+q^{-1}}\theta^\diamond_{M'}\right) ,$$

$$\frac{\langle\theta_M|\Psi_+^{M''}(\bar{u},m)\rangle}{\langle\theta_M|\theta_M\rangle} = \mathcal{G}^+_{M''}(\bar{u})\frac{f_M}{g_0\nu_0^{\{*,\diamond,.\}}}\sum_{M'=0}^{2s}\frac{1}{h_{M'}}(P^{\{\diamond,.\}^{-1}})_{MM'}\prod_{i=1}^{M''}\left(U_i-\frac{r_0}{q+q^{-1}}\theta^\diamond_{M'}\right) .$$

Use properties of Leonard triple $A, A^*, A^\diamond \Rightarrow$ **Scalar products** $\langle \theta_M | \Psi_\epsilon^{M'}(\bar{u}) \rangle$ (Dual basis $\langle \theta_M |$ linear maps $\mathcal{V} \to \mathbb{C}$)

Theorem : scalar products for arbitrary 'off-shell' Bethe states

$$\frac{\langle \theta_M | \Psi_-^{M''}(\bar{u}, m) \rangle}{\langle \theta_M | \theta_M \rangle} = \mathcal{G}_{M''}^-(\bar{u}) \frac{f_M}{f_0} \sum_{M'=0}^{2s} (P^{\{\diamond,.\}^{-1}})_{MM'} \prod_{i=1}^{M''} \left(U_i - \frac{r_0}{q+q^{-1}} \theta_{M'}^\diamond \right) ,$$

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Example : $s = \frac{1}{2}, \epsilon = -$

$$X_M^{\pm,off}(u_1) = \frac{\langle \theta_M | \Psi_\pm^1(u_1, m) \rangle}{\langle \theta_M | \theta_M \rangle} , \quad M = 0, 1 .$$

$$X_0^{-,off}(u_1) = \frac{q^3 r_0^2 \mathsf{b}^2}{b(q)(q^2 r_0^2 \mathsf{b}^2; q^2)_1} u_1 b(u_1^2) Y_-(u_1|u_1)$$

with
$$Y_-(x|x) = \frac{b(q)}{q^3 r_0^3 \mathsf{b}\mathsf{b}^* \mathsf{b}^\diamond} \left(\frac{q r_0 \mathsf{b}^\diamond \mathsf{b}^* \left(q^2 r_0^2 \mathsf{b}^2 - 1 \right)}{x^2} + q^3 x^2 r_0 \mathsf{b}^\diamond \mathsf{b}^* \left(q^2 r_0^2 \mathsf{b}^2 - 1 \right) + \mathsf{b}^* \left(1 - q^4 r_0^2 \mathsf{b}^2 \right) \right.$$
$$\left. - \left(q^2 - 1 \right) q r_0 \mathsf{b}\mathsf{b}^\diamond \left(q^2 r_0^2 \left(\mathsf{b}^* \right)^2 + 1 \right) + q^2 r_0^2 \left(\mathsf{b}^\diamond \right)^2 \mathsf{b}^* \left(1 - q^4 r_0^2 \mathsf{b}^2 \right) \right) .$$

Step 7 : Specialization and Belliard-Slavnov linear system

Use Leonard triple $\Rightarrow$ Scalar products $\langle\theta_M|\Psi_\epsilon^{M'}(\bar{u})\rangle$ —Specialization $M' = 2s$→ Belliard-Slavnov linear system solved by $\langle\theta_M|\Psi_\epsilon^{2s}(\bar{u})\rangle$

$\to$ In general **ABA setting** [Belliard-Slavnov,'19] showed that **scalar product of Bethe states** satisfy a system of linear equations.

$\Rightarrow$ gives **a way to derive determinant formula for scalar products**

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Belliard-Slavnov linear system for LP A, A^* :

Proposition : For any $M=0,1,...,2s$, and $j=1,2,...,2s+1$,

$$X_M^{\epsilon,off}(\bar{u}) = \frac{\langle\theta_M|\Psi_\epsilon^{2s}(\bar{u},m)\rangle}{\langle\theta_M|\theta_M\rangle} \qquad \text{solves} \qquad \sum_{k=1}^{2s+1} \mathcal{M}_{jk}^{\epsilon,M} X_M^{\epsilon,off}(\bar{\mathcal{Y}}_k) = 0$$

- Matrices $\mathcal{M}^{\epsilon,M}$, $M=0,1,...,2s, \epsilon=\pm$ with entries $\mathcal{M}_{jk}^{\epsilon,M}$ with $j,k=1,...,2s+1$ expressed in terms of **ABA data**
- $\bar{\mathcal{Y}} = \{y_1, y_2, ..., y_{2s}, y_{2s+1}\}$, $\quad \bar{\mathcal{Y}}_j = \bar{\mathcal{Y}} \backslash y_j \quad$ and $\quad \bar{\mathcal{Y}}_{j,k} = \bar{\mathcal{Y}}_j \backslash y_k$.

Use Leonard triple $\Rightarrow$ Scalar products $\langle\theta_M|\Psi_\epsilon^{M'}(\bar{u})\rangle$ —Specialization $M'=2s$→ Belliard-Slavnov linear system solved by $\langle\theta_M|\Psi_\epsilon^{2s}(\bar{u})\rangle$

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Using theory of Leonard pairs and linear independence of Bethe states, one shows :

Lemma : $rank(\mathcal{M}^{\epsilon,M}) = 2s$.

$\Rightarrow \exists$ **Determinant formula** for $\frac{\langle\theta_M|\Psi_\epsilon^{2s}(\bar{u},m)\rangle}{\langle\theta_M|\theta_M\rangle}$

q-Racah polynomials $R_M(\theta^*_N)$
$\exists$ Determinant formula

Belliard-Slavnov linear system
solved by $\langle \theta_M | \Psi^{2s}_\epsilon(\bar{u}) \rangle$

Proposition : The **normalized scalar product** admits a determinant formula :

$$\frac{\langle \theta_M | \Psi^{2s}_\epsilon(\bar{u}, m) \rangle}{\langle \theta_M | \theta_M \rangle} = \psi^\epsilon_M (-1)^k \det \widetilde{\mathcal{M}}^{\epsilon,M}_{[2s+1,k]} \,, \qquad \widetilde{\mathcal{M}}^{\epsilon,M} = \mathcal{W}^{\epsilon,M} \mathcal{M}^{\epsilon,M}$$

for some non-degenerate $2s+1 \times 2s+1$ matrix $\mathcal{W}^{\epsilon,\mathcal{M}}$ such that :

$$(i) \qquad \left(\mathcal{W}^{\epsilon,M} \mathcal{M}^{\epsilon,M} \right)_{2s+1 \ k} = 0 \qquad \text{for all} \quad k = 1, ..., 2s+1 \,,$$

$$(ii) \qquad \det \widetilde{\mathcal{M}}^{\epsilon,M}_{[2s+1,k]} \qquad \text{is independent of } y_k .$$

$\rightarrow$ Application to q-Racah polynomials ? If we specialize $\bar{u}$ to Bethe roots such that $|\Psi^{2s}_\epsilon(\bar{u}, m)\rangle \propto |\theta^*_N\rangle$

$\Rightarrow$ building elements for q-Racah polynomials....

Step 8 : Determinant formula and q-Racah polynomials

q-Racah polynomials $R_M(\theta^*_N)$
$\exists$ Determinant formula

Belliard-Slavnov linear system
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Corollary : The q-Racah polynomial $R_M^{\{\cdot,\diamond\}}(\theta^*_N)$ admits a **determinant formula** :

$$R_M^{\{\cdot,\diamond\}}(\theta^*_N) = \frac{\psi^-_M}{\psi^-_0} \frac{\det \widetilde{\mathcal{M}}^{-,M}_{[2s+1,2s+1]}}{\det \widetilde{\mathcal{M}}^{-,0}_{[2s+1,2s+1]}} \qquad \text{for} \quad \underbrace{\bar{\mathcal{Y}}_{2s+1} = S^{*N(i)}_-}_{\text{solve BEs}} .$$

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Example : $s = \frac{1}{2}$ $\qquad R_0^{\{\cdot,\diamond\}}(\theta^*_0) = R_0^{\{\cdot,\diamond\}}(\theta^*_1) = 1$ and

$$R_1^{\{\cdot,\diamond\}}(\theta^*_N) = \frac{\psi^-_1}{\psi^-_0}\frac{\det \widetilde{\mathcal{M}}^{-,1}_{[2,2]}}{\det \widetilde{\mathcal{M}}^{-,0}_{[2,2]}} = \frac{\psi^-_1}{\psi^-_0}\frac{\widetilde{\mathcal{M}}^{-,1}_{11}}{\widetilde{\mathcal{M}}^{-,0}_{11}}$$

$$= -\frac{b(q)^2\left(-\frac{qr_0\mathsf{b}\mathsf{b}^\diamond}{\mathsf{b}^*};q^2\right)_1\left(-\frac{qr_0\mathsf{b}^\diamond\mathsf{b}^*}{\mathsf{b}};q^2\right)_1}{q^2r_0^3\mathsf{b}\mathsf{b}^\diamond}\overbrace{\frac{1}{Y_-(y_1|y_1)}}^{\text{ABA datum}}, \quad y_1 \text{ solves BE}.$$

Summary : **New insights in both subjects**

- **Scalar products of on-shell/off-shell Bethe states** are explicitly computed for Leonard pairs' related related integrable systems ;
- Explicit solutions of **Belliard-Slavnov linear systems** are obtained using theory of Leonard triple ;
- The existence of a **determinant formula for q-Racah polynomials** is proven. Explicit examples are constructed.

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- The existence of a **determinant formula for q-Racah polynomials** is proven. Explicit examples are constructed.

Perspectives : $\rightarrow$ **Extend** to $q = 1$ **Racah case** based on [Bernard et al.,'21-'23]

$\rightarrow$ **Extend** to **tridiagonal pairs of q-Racah type**

The q-**Onsager algebra** O_q admits 'alternating' presentation with generators $W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1}$ [B-Shigechi,'09],[B-Belliard,'17],[Terwilliger,'21]

- Let $\pi : O_q \rightarrow \mathrm{End}(\mathcal{V})$. **Diagonalization** of $\pi(W_{-k})$ (or $\pi(W_{k+1})$) using **ABA**
  $\pi(W_0) = A$, $\pi(W_1) = A^*$ **tridiagonal pair of q-Racah type**
- **Scalar products of Bethe states**
  $\Rightarrow$ **Transition matrix** between $\pi(W_{-k})$ and $\pi(W_{k+1})$'s eigenbases

THANK YOU FOR YOUR ATTENTION!!!

→ Consider the image in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ of AW generators A, A* [Granovskii-Zhedanov,'93]. The image of $\mathsf{H}(\lambda) = \mathsf{A} + \lambda \mathsf{A}^*$ is :

$$H(\lambda) = S_1^x S_2^x + S_1^y S_2^y + \Delta S_1^z S_2^z + \lambda (S_2^x S_3^x + S_2^y S_3^y + \Delta S_2^z S_3^z) + \frac{1}{q-q^{-1}}(S_1^z + (\lambda-1)S_2^z - \lambda S_3^z) , \qquad \Delta = \frac{q+q^{-1}}{2} ,$$

where

$$S^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad S^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

→ Consider finite. dim. irreps of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and corresponding Leonard pair A, A^* [Huang,'16]

- Eigenstates for $\lambda \neq 0$ of inhomogeneous type : $|\Psi_+^{2s}(\bar{u})\rangle$;
- Eigenstates for $\lambda = 0$: $|\theta_M\rangle$;
- The **normalized scalar product** $\frac{\langle \theta_M | \Psi_+^{2s}(\bar{u},m)\rangle}{\langle \theta_M | \theta_M \rangle}$ determines the overlap between the eigenstates of $H(\lambda)$ and eigenstates of H'.
- Application : in the context of quantum phase transitions λ called 'driving parameter' ⇒ **fidelity** when considering the **ground states** of each Hamiltonian.