From cyclotomic to orthogonal polynomials: Sturm meets Ramanujan

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dedicated to Paul Terwilliger on the occasion of his 70th birthday

Motivation

1. Study inverse Sturm problem for polynomials orthogonal on the real line and on the unit circle

2. Relate objects from abstract algebra and number theory (cyclotomic polynomals, Ramanujan's trigonometric sums) to OPRL and OPUC

3. Construct new explicit families of OPRL and OPUC

4. Apply Sylvester duality property to derive some basic results

Key publications:

(i) AZ, *Classical Sturmian sequences*, **arXiv:1904.03789**.

(ii) AZ, *Para-orthogonal polynomials on the unit circle generated by Kronecker polynomials*, **arXiv:2107.11430**.

(iii) AZ, *Ramanujan's trigonometric sums and para-orthogonal polynomials on the unit circle* , **arXiv:2107.12543**.

Starting point: inverse Sturm problem on the real line

Classical Sturm problem.

Polynomial $P_{N+1}(x) = x^{N+1} + a_N x^N + \cdots + a_1 x + a_0$ with **prescribed real** coefficients a_i. Choose $P_N(x) = (N+1)^{-1} P'_N(x)$ and compute sequence of monic polynomials by Euclidean algorithm.
First step

$$P_{N+1}(x) = q_N(x) P_N(x) - u_N P_{N-1}(x), \quad \deg(q_N(x)) = 1$$

Further steps

$$P_{n+1}(x) = q_n(x) P_n(x) + u_n P_{n-1}(x), \quad n = N, N-1, N-2, \dots,$$

$$\deg(P_n(x)) \le n, \quad \deg(q_n(x)) \ge 1$$

Final polynomial $P_{N+1-M}(x) = 1$, M - number of members of Sturm sequence. Use sequence $P_{N+1}(x), P_N(x), \dots, P_{N+1-M}(x)$ to find number of zeros of $P_{N+1}(x)$ on $[a, b]$.

Inverse Sturm problem.

Start with **prescribed zeros** of $P_{N+1}(x)$. What are properties of Sturmian sequence?

If all zeros $x_0, x_1, \dots, x_N$ of $P_{N+1}(x)$ are **real and simple** then Sturmian sequence $P_0(x) = 1, P_1(x), \dots, P_{N+1}(x)$ is a set of fiinite orthogonal polynomials:

$$\sum_{s=0}^{N} w_s P_n(x_s) P_m(x_s) = h_n \delta_{nm}$$

weights

$$w_s = \frac{h_N}{P'_{N+1}(x_s)^2}$$

Recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \quad \deg(P_n(x)) = n$$

What are **explicit** polynomials $P_n(x)$ if one start with "classical" grid x_s?

Simplest example - **uniform grid**

$$x_s = s,\ s = 0, 1, 2, \ldots N$$

Corresponding Sturmian sequence - Hahn polynomials

$$P_n(x) = H_n(x, \alpha, \beta, N), \quad \alpha = \beta = -N - 1$$

Similarly, for **quadratic grid** $x_s = s(s+1)$ the Sturmian sequence is a special case of Racah polynomials (AZ, 2019)

Mirror (Sylvester) duality

Let $P_n(x)$ be the Sturmian chain on prescribed grid x_s with recurrence coefficients b_n, u_n.

Let $P_n^*(x)$ be "mirror-dual" orthogonal polynomials with recurrence coefficients b_{N-n}, u_{N+1-n}.

Then $P_n^*(x)$ are of **Legendre type** with **equal weights** $w_s = 1, \ s = 0, 1, 2, \dots, N$

$$\sum_{s=0}^{N} P_n(x_s) P_m(x_s) = h_n^* \delta_{nm}$$

In implicit form (i.e. for continued fractions, not for OP) this result goes back to Sylvester (1853).

Examples

For uniform grid $x_s = 0, 1, 2, ...N$ the Legendre-type polynomials are special case of **Hahn polynomials**

$$P_n^*(x) = H_n(x; \alpha, \beta, N)$$

with $\alpha = \beta = 0$. Discovered by Chebyshev (1864).

$$P_n^*(x) = {}_3F_2 \left(\begin{matrix} -n, -x, n+1 \\ -N, 1 \end{matrix} ; 1 \right)$$

For quadratic grid $x_s = s(s+1)$ the Legendre-type polynomials are special case of **Racah polynomials** (AZ, 2019).

Sylvester triple

Start with given points $x_0, x_1, \dots, x_N$. Characteristic polynomials $P_{N+1}(x) = \prod_{i=0}^{N}(x - x_i)$.

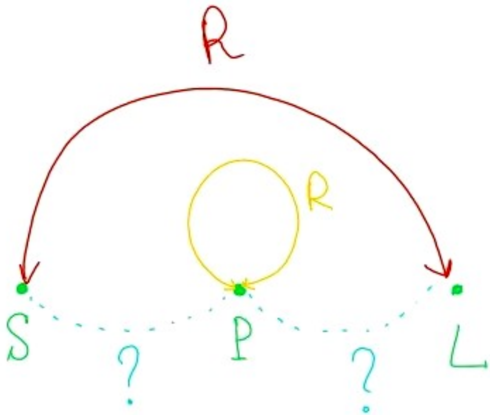
Consider the weight functions depending on integer parameter k

$$w_s^{(k)} = \left|P'_{N+1}(x_s)\right|^{-k}, \; s = 0, 1, 2, \dots, N$$

Then we have the **Sylvester triple** of polynomials:

1. weight $w_s^{(0)} = 1$ corresponds to Legendre case (equal weights) = type L
2. weight $w_s^{(1)}$ corresponds to persymmetric Jacobi matrices of orthogonal polynomials, i.e. $u_{N+1-n} = u_n, b_{N-n} = b_n$ = type P
3. weight $w_s^{(2)}$ corresponds to Sturmian orthogonal polynomials = type S

Sylvester triple graph



Jacobi matrices for types L and and S are dual with respect to antidiagonal (reflection) matrix. Jacobi matrix for the polynomials of type P are self-dual with respect to this symmetry (persymmetric).

What are explicit 3 families of Sylvester triple for some known grids x_s on **real** line?
Examples:

- linear grid; P type - Krawtchouk, L and S types - two special Hahn polynomials
- quadratic grid; P type - special dual Hahn, L and S types - special Racah
- q-quadratic grid; P type - special q-Racah, L and S types - unknown
- bi-lattice; P type - para-Krawtchouk, L and S types unknown
- Sylvetser triples for other grids???

Non-real zeros

If some of zeros x_s of $P_{N+1}(x)$ are non-real then the Sturmian sequence $P_n(x)$ doesnot belong to OP.

Example

Take

$$P_{N+1}(x) = \frac{x^{N+2} - 1}{x - 1} = x^{N+1} + x^N + \cdots + x + 1$$

Zeros - roots of unity

$$x_s = \exp\left(\frac{2\pi i(s+1)}{N+2}\right), \ s = 0, 1, \dots, N$$

Sturmian sequence consists of only **four** members:

$$P_{N+1}(x), P_N(x) = (N+1)^{-1} P_{N+1}(x),$$

$$P_{N-1}(x) = x^{N-1} + 2x^{N-2} + 3x^{N-3} + \cdots + (N-1)x + N, \quad P_{N-2}(x) = 1$$

Complete Sturm sequences with non-real zeros

The Sturm sequence is **complete** if $deg(P_n(x)) = n$ and
$n = N, N-1, N-2, \dots, 1, 0.$
Equivalently, the complete Sturm sequence consists of $N+2$ members

$$P_{N+1}(x), P_N(x), P_{N-1}(x), \dots, P_0(x) = 1$$

Then $P_k(k)$, $k = 0, 1, \dots P_N(x)$ is a sequence of **formal** orthogonal polynomials with rr

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x),$$

where $u_n \neq 0$ but NOT necessary $u_n > 0$.

In general, if roots of $P_{N+1}(x)$ are not real then the Sturm sequence is **incomplete**

Example of complete Sturm sequence with complex zeros

Consider

$$P_{N+1}(x) = \left(x^2 + (1/2)^2\right)\left(x^2 + (3/2)^2\right)\dots\left(x^2 + (N/2)^2\right)$$

where $N = 1, 3, 5, \dots$. All zeros are complex

$$x_s = \pm \left(\frac{k}{2}\right) i, \; k = 1, 3, \dots, N$$

Nevertheless, teh Sturm sequence is **complete**. Coincides with special Hahn polynomials $H_n(ix, \alpha, \alpha; N)$

Complete Sturm sequences of cyclotomic polynomials

Cyclotomic polynomial

$$C_N(x) = \prod_{k=1}^{J}(x - \tau_k) = \prod_{k=1}^{J}(x - \zeta^k),$$

where τ_k are all **primitive** N-th roots of unity and ζ is one of these roots

$$\zeta = \exp\left(\frac{2\pi j}{N} i\right)$$

with j coprime with N.

$$J = \varphi(N)$$

is totient Euler function.

Example. $N = 6$. Primitive roots $1, 5$. $C_6(x) = (x - \zeta)(x - \zeta^5) = x^2 - x + 1$

Properties of cyclotomic polynomials

- C_n monic polynomials with integer coefficients
- even degree, $C_n(0) = 1$ for all $n > 1$
- irreducible over field $\mathbb{Q}$
- $C_p(x) = x^{p-1} + x^{p-2} + \cdots + 1$
- $C_{2p}(x) = C_p(-x)$
- if $n = rp^m$ then $C_n(x) = C_{pr}\left(x^{p^{m-1}}\right)$
- if $n = p_1 p_2$ then coefficients of $C_n(x)$ are $0, 1, -1$

Mirror duals and Ramanujan sums

Assume that Sturmian sequence $P_0, P_1(x), \dots, P_N(x), P_{N+1}(x) = C_M(x)$ of a cyclotomic polynomial $C_M(x)$ is **complete**. In this case $N = \varphi(M)$. Then mirror dual sequence $P_0, P_1^*(x), \dots, P_N^*(x), P_{N+1}^*(x) = C_M(x)$ is **complete** as well.

Orthognality for dual sequence

$$\sum_{s=1}^{N} P_n^*(\tau_s) P_m^*(\tau_s) = h_n^* \delta_{nm}$$

Moments for duals

$$c_n^* = \sum_{s=1}^{N} \tau_s^n = \sum_{(s,M)=1} \exp\left(\frac{2\pi i s n}{M}\right) = c_M(n)$$

$c_M(n)$ - trigonometric Ramanujan sum.

Conditions of complete Sturm sequence

Sturm sequence is complete iff $u_n \neq 0,\ n = 1, 2, \ldots, N$. Equivalently iff $u_n^* \neq 0,\ n = 1, 2, \ldots, N$ because $u_n^* = u_{N+1-n}$.
Equivalently iff

$$\Delta_n^* \neq 0,\ n = 1, 2, \ldots, N$$

where **Hankel determinants**

$$\Delta_n^* = \begin{vmatrix} c_M(0) & c_M(1) & \ldots & c_M(n-1) \\ c_M(1) & c_M(2) & \ldots & c_M(n) \\ \ldots & \ldots & \ldots & \ldots \\ c_M(n-1) & c_M(n) & \ldots & c_M(2n-2) \end{vmatrix}, \quad n = 1, 2, \ldots$$

Theorem. Sturm sequence of cyclotomic polynomial $C_M(x)$ is complete iff Hankel determinants of Ramanujan sums are nonzero: $\Delta_n^* \neq 0,\ n = 1, 2, \ldots, N$

First 100 M

Among first 100 only the following cyclotomic polynomials $C_M(x)$ produce complete Sturm sequences:

$$M = 2, 3, 4, 6, 12, 15, 30, 60$$

What about recurrence coefficients?

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = 0$$

For $M = 15$ $N = 8$ and

$$u_1 = -\frac{5}{4},\ u_2 = \frac{24}{25},\ u_3 = -\frac{25}{144},\ u_4 = -\frac{216}{25},\ u_5 = \frac{35}{36},\ u_6 = -\frac{240}{49},\ u_7 = \frac{7}{64}$$

and

$$b_0 = \frac{3}{2}, b_1 = -\frac{7}{10}, b_2 = \frac{7}{60}, b_3 = \frac{137}{60}, b_4 = -\frac{71}{30}, b_5 = \frac{109}{42}, b_6 = -\frac{143}{56}, b_7 = \frac{1}{8}.$$

Examples of Hankel determinants

Case $M = 15 = 3 \cdot 5$ $(N = 8)$ is complete. Hankel sequence is

$$\Delta_1 = 8, \ \Delta_2 = 7, \ \Delta_3 = -30, \ \Delta_4 = 125, \ \Delta_5 = 4500,$$

$$\Delta_6 = -28125, \ \Delta_7 = 168750, \ \Delta_8 = 1265625$$

Case $M = 21 = 3 \cdot 7$ $(N = 12)$ is **incomplete**. Hankel sequence is

$$\Delta_1 = 12, \ \Delta_2 = 11, \ \Delta_3 = -42, \ \Delta_4 = 0.$$

Main conjecture

For bigger M complete Sturm sequences become more rare. E.g. for $100 \leq M \leq 200$ there are only 3: $M = 105, 165, 195$. For $200 \leq M \leq 1000$ there are 7: $M = 210, 255, 330, 390, 420, 510, 660, 780$.

Experimentally supposed the complete Sturm sequences:

- finite **ternary** sequence $M_K = 3 \cdot 5 \cdot 7,\ 3 \cdot 5 \cdot 11,\ 3 \cdot 5 \cdot 13,\ 3 \cdot 5 \cdot 17$
- **principal** infinite sequence $M_K = p_1 p_2 \dots p_K$ of **odd** primes , i.e. $M = 1, 3,\ 3 \cdot 5,\ 3 \cdot 5 \cdot 7,\ 3 \cdot 5 \cdot 7 \cdot 11, \dots$
- infinite **quaternary** sequence $M_K = 3 \cdot 5 \cdot 7 \cdot p_K$, where $p_K > 7$ i.e. $M_K = 3 \cdot 5 \cdot 7 \cdot 11,\ 3 \cdot 5 \cdot 7 \cdot 13,\ 3 \cdot 5 \cdot 7 \cdot 17, \dots$
- infinite **quaternary** sequence $M_K = 3 \cdot 5 \cdot 11 \cdot p_K$, where $p_K > 11$
- infinite **quaternary** sequence $M_K = 3 \cdot 5 \cdot 13 \cdot p_K$, where $p_K > 13$
- infinite **quaternary** sequence $M_K = 3 \cdot 5 \cdot 17 \cdot p_K$, where $p_K > 17$
- $\dots ??? \dots$
- all previous M_K multiplied by 2 or 4, i.e. $2M_K$ and $4M_K$

Sturmian sequence for orthogonal polynomials on the unit circle

What is an analog of Sturm problem for the unit circle? Let $\zeta_0, \zeta_1, \dots, \zeta_N$ be distinct points on unit circle $|\zeta_i| = 1$.
Define

$$\Phi_{N+1}(z) = (z - \zeta_0)(z - \zeta_1)\dots(z - \zeta_N)$$

and

$$\Phi_N(z) = (N+1)\Phi'_{N+1}(z)$$

By Gauss-Lucas theorem, all zeros of $\Phi_N(z)$ lie inside unite circle.

Then one can define the Sturmian sequence $\Phi_n(z), n = N-1, N-2, \dots, 0$ by Szegő formula

$$\Phi_n(z) = \frac{\Phi_{n+1}(z) + \bar{a}_n \Phi^*_{n+1}(z)}{z(1 - |a_n|^2)}$$

where

$$\Phi_n^*(z) = z^n \bar{\Phi}_n(1/z), \quad a_n = -\bar{\Phi}_{n+1}(0)$$

It is easy to show that $|a_n| < 1$ for $n = 0, 1, \ldots, N-1$ and $|a_N| = 1$.

The sequence $\Phi_n(z)$ is **unique**. Szegő recurrence

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{a}_n \Phi_n^*(z)$$

means that $\Phi_n(z)$ is a finite sequence of polynomials orthogonal on the unit circle (OPUC).

$$\sum_{s=0}^{N} w_s \Phi_n(\zeta_s) \bar{\Phi}_m(\zeta_s^{-1}) = h_n \delta_{nm}$$

weights

$$w_s = \frac{h_N}{\left|\Phi'_{N+1}(\zeta_s)\right|^2} > 0$$

very similar to real Sturmian sequence!

Normalization coefficients are expressed in terms of Verblunsky parameters

$$h_n = (1 - |a_0|^2)(1 - |a_1|^2)\dots(1 - |a_{n-1}|^2) > 0$$

What are possible "nice" grids for the points $\zeta_0, \zeta_1, \dots, \zeta_N$?

Main idea: use **Kronecker polynomials**

Kronecker polynomials

Kronecker in 1857 defined a special class of monic polynomials

$$K(z) = z^n + A_{n-1}z^{n-1} + A_{n-2}z^{n-2} + \dots A_1 z + A_0$$

with **integer** coefficients A_i and with condition $A_0 \neq 0$. Main condition: all zeros of $K(z)$ should lie in the unit disk $|z| \le 1$.

Then it is trivially seen that all zeros belong to unit circle $|\zeta_i| = 1$:

$$A_0 = (-1)^n \zeta_0 \zeta_1 \dots \zeta_{n-1}$$

Hence $|A_0| \le 1$. But A_0 is integer. Hence $A_0 = \pm 1$. And for all zeros $|\zeta_i| = 1$

Less trivial property (Kronecker): all zeros are roots of unity. Explicit formula

$$K(z) = C_{m_1}^{j_1}(z) C_{m_2}^{j_2}(z) \dots C_{m_k}^{j_k}(z)$$

where $C_k(z)$ are **cyclotomic** polynomials.

Simplest examples

(i) Trivial $\Phi_{N+1}(z) = z^{N+1} - 1$. All zeros are N-th roots of unity

$$\Phi_n(z) = z^n, \quad n = 0, 1, \dots, N$$

This is "free" system of OPUC. All Verblunsky parameters are zero: $a_n = 0$ apart from $a_N = 1$.

(ii) Less trivial $\Phi_{N+1}(z) = \frac{z^{N+1}-1}{z-1}$. All zeros are N-th roots of unity apart from $z = 1$.
"Single momentum" OPUC (Simon)

$$\Phi_n(z) = \frac{1}{n+1} \sum_{k=0}^{n} (k+1) z^k, \quad n = N, N-1, \dots, 0$$

with

$$a_n = -\frac{1}{n+2}, \; n = 0, 1, 2, \dots, N-1, \; a_N = -1.$$

Toeplitz determinants

$$\Delta_n = \begin{vmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_n \\ \sigma_{-1} & \sigma_0 & \dots & \sigma_{n-2} \\ \dots & \dots & \dots & \dots \\ \sigma_{-n} & \sigma_{1-n} & \dots & \sigma_0 \end{vmatrix}, \quad n = 1, 2, \dots$$

Main property:

$$h_n = 1 - |a_n|^2 = \frac{\Delta_n}{\Delta_{n-1}} > 0$$

Hence condition $\Delta_n > 0$, $n = 0, 1, \dots,$ is equivalent to condition $|a_n| < 1$, $n = 0, 1, \dots, N-1$.

Main theorem for Toeplitz determinants of RS: **for every** M

$$\begin{vmatrix} c_M(0) & c_M(1) & \dots & c_M(n) \\ c_M(1) & c_M(0) & \dots & c_M(n-1) \\ \dots & \dots & \dots & \dots \\ c_M(n) & c_M(n-1) & \dots & c_M(0) \end{vmatrix} > 0, \quad n = 1, 2, \dots, M-1$$

Open problems

- Find explicit expressions for binary and ternary cyclotomic Sturmian OPUC
- Find more explicit examples of Sturmian Kronecker OPUC
- Find limiting cyclotomic and Kronecker OPUC when $N \to \infty$
- Find polynomials on the real line by Szegő map of Sturmian Kronecker OPUC
- Possible bispectrality of Sturmian (Legendre) OPUC