Compact symmetric spaces (Compact Gelfand pairs) vs. finite Gelfand pairs

Eiichi Bannai

Professor Emeritus of Kyushu University

TerwilligerFest

at Kranjska Gora in Slovenia.

June 26, 2025

This is a preliminary report of ongoing joint work with Hirotake Kurihara (Yamaguchi University) and Da Zhao (East China University of Science and Technology). I also expect the involvement of some other researchers to this project.

I would like to apologize that I will give more speculations and personal views, rather than presenting rigorous and completed results. Also, I apologize that I may have to use many terminologies without giving the exact definitions. I will mainly talk on the following three topics.

(i) Commutative association schemes and finite Gelfand pairs. (Note that a finite Gelfand pair means a Schurian commutative association scheme, and is also equivalent to a multiplicity-free finite permutation groups.) In addition, we recall P-polynomial, Q-polynomial, and P-and Q-polynomial association schemes, as well as the concept of <u>multivariate version</u> of P- and/or Q-polynomial association schemes.

(ii) We discuss for each compact symmetric space, are there any finite Gelfand pairs naturally attached to it?

(iii) How we can approach the classification of general finite Gelfand pairs, in particular, toward the classification of primitive P-and Q-polynomial finite Gelfand pairs?

The classification problem of P-and Q-polynomial association schemes has been (and still is) a central problem in algebraic combinatorics, since around 1980.

It is known that the spherical functions (and the character table) of P-and Q-polynomial association schemes are described by Askey-Wilson orthogonal polynomials (of one variable), including their special cases and limiting cases. See D. Leonard (SIAM J. Math. Anal.,1982). See also Bannai-Ito's book: Algebraic Combinatorics, I (Benjamin/Cummings, 1984) and the new book: Algebraic Combinatorics (De Gruyter, 2021) by Bannai-Bannai-Ito-Tanaka, and also the papers by Terwilliger (LAA, 2001 and Graphs and Comb., 2021).

The recent study of multivariate P-polynomial (and Q-polynomial) association scheme was first started by Bernard–Crampé-d'Andecy-Vinet-Zaimi (Alg. Comb., 2024). Then the concept was extended by Bannai-Kurihara-Zhao-Zhu (JCT(A),2025).

The orthogonal polynomials appearing here are multivariable orthogonal polynomials.

Let us recall that, roughly speaking, compact symmetric spaces were classified by E. Cartan (1926). Compact symmetric spaces are all Gelfand pairs. There is a weaker concept called "weakly symmetric spaces" defined by Selberg (1956) and are also shown to be Gelfand pairs. Also, weakly symmetric spaces are classified (Akhiezer-Vinberg, 1999; Nguyen, 2000). On the other hand, under some mild additional conditions such as compact, connected, G being a simple Lie group, the compact Gelfand pairs were already classified (cf. Krämer, 1979).

So, we basically have the list of compact Gelfand pairs (for Lie groups). We were not aware of this fact until very recently.

We first remark that it seems that any compact symmetric space of rank ℓ (if necessary, under the additional assumption of simply connectedness) has the property of multivariate (ℓ-variate) Q-polynomial property similar to the concept of multivariate (ℓ-variate) Q-polynomial property in our sense. This claim is obtained from the work of Vretare (1976) where the spherical functions of the compact symmetric spaces were obtained. In passing, it seems that these compact symmetric spaces also have the property similar to ℓ-variate P-polynomial property. This claim is obtained by using the concept of invariant differential operators (on the symmetric space) which corresponds to adjacency matrices A_i in the Bose-Mesner algebra (=Hecke algebra) in the (finite) association scheme situation. So, the compact symmetric spaces can be regarded as to have the multivariate (ℓ-variate) P-and Q-polynomial property. To know this was a real surprise for us, because this means that the compact symmetric spaces (so almost all the compact Gelfand pairs) have the property of the multivariate (ℓ-variate) P-and Q-polynomial property.

The classification of irreducible compact symmetric spaces is due to E. Cartan. If we exclude the cases where Lie group itself become so, there are the following 7 series (for the cases where infinite family appear). Here we describe only simply connected ones.)
AI $SU(n)/SO(n)$,
AII $SU(2n)/Sp(n)$,
AIII $SU(p+q)/S(U(p)\times U(q))$,
BDI $SO(p+q)/(SO(p)\times SO(q))$,
DIII $SO(2n)/U(n)$,
CI $Sp(n)/U(n)$,
CII $Sp(p+q)/(Sp(p)\times Sp(q))$.

The cases AIII, BDI, CII are Grassmannian spaces. Let us discuss for other 4 remaining cases AI, AII, DIII, CI in the first place.

• To AI $SU(n)/SO(n)$, $GL(n,q^2)/GL(n,q)$ is a corresponding finite Gelfand pair (see Gow (1984).)

• To AII $SU(2n)/Sp(n)$, $GL(2n,q)/Sp(2n,q)$ is a corresponding finite Gelfand pair (see Kliyachko (1984) and Bannai-Kawanaka-Song (1990).)

• To DIII $SO(2n)/U(n)$, $GL(2n,q)/GL(n,q^2)$ is a corresponding finite Gelfand pair (see Bannai-Tanaka (2003) and Henderson (2003).)

• To CI $Sp(n)/U(n)$, $Sp(2n,q)/GU(n,q)$ is a strong candidate of the finite version of the Siegel upper half plane $Sp(n)/U(n)$. (This is due to Pantoja, Soto-Andrade and Vargas: On the construction of a finite Siegel space (J. Lie Theory, 2015).) Their claim is very convincing.

On the other hand, it is very delicate whether $Sp(2n,q)/GU(n,q)$ becomes a Gelfand pair or not. In fact we showed that there are some special cases (perhaps this is the general case) that we cannot get any Gelfand pair attached to $Sp(2n,q)/GU(n,q)$ even if we consider the additional action of automorphism.)

For the cases of Grassmannian spaces AIII, BDI, CII, what should be their finite versions? The cases of acting on isotropic subspaces are very much studied and well known. On the other hand, actions on non-isotropic subspaces seems to be more natural and appropriate, but it seems difficult to determine when they are actually multiplicity-free. So, I would like to propose the following problem.

Problem 1. Let us consider a classical form on the vector space over a finite field. If we consider the transitive action on the spaces of non-singular k-dim subspaces, how much can the multiplicities of irreducible representations become large? Is there any good upper bound? Is it bounded by a function depending only on k?

Also, we are interested in the following problem.

Problem 2. How large the multiplicities of the irreducible representations appearing in the permutation character of $Sp(2n,q)/GU(n,q)$ can become? How close they are to be a Gelfand pair, although they are not Gelfand pairs in general.

There are some works try to study finite version of compact symmetric spaces.

Finite symmetric spaces. (Cf. G. Lusztig: Symmetric spaces over a finite field (1990).)

Quandles. (Cf. D. Joyce: Simple quandles, J. Alg. (1982).)

But we will not discuss these today. Instead, Now, we want to discuss the classification problem of finite Gelfand pairs (G, K), in particular, of finite primitive Gelfand pairs.

The classification of maximal subgroups, in particular for finite (almost) simple groups has been developed extensively using the classification of finite simple groups. Notably, the O'Nan-Scott theorem and the work of Aschbacher: On the maximal subgroups of the finite classsical groups (1984). (These results are very involved and very difficult. Here, we briefly review this situation.

We can use the classification of finite simple groups extensively for the study of Gelfand pairs. One model example of the use of classification of finite simple groups is the following theorem. (Praeger-Saxl-Yokoyama (1987). For a primitive distance transitive graph and group (i.e., for a primitive P-polynomial Gelfand pair (G, H)) only the following three cases must occur:
(i) The graph is the Hamming graph.
(ii) Almost simple case. I.e., there is a nonabelian simple group G_0 such that $G_0 \leq G \leq Aut(G_0)$.
(iii) Affine case. I.e., there is a regular normal subgroup N (of G) that is an elementary abelian p group.

It seems that the case (iii) is now settled. Cf. survey paper of van Bon (Europ. J. Comb., 2015). So, only the case (ii) Almost simple case, is open. The case of G_0 being alternating group and sporadic simple group cases are settled. So, the case of classical or exceptional type Lie type simple group case are open. It seems that the situation is close to the goal, but it seems that it is not yet settled completely.

Toward the classification of primitive finite Gelfand pairs.

We are interested in knowing the following objects.

(i) Maximal subgroups of a finite group.
(ii) Multiplicity-free maximal subgroups of a finite group (= finite primitive Gelfand pairs).
(iii) P-polynomial maximal subgroups of a finite group. (Distance transitive groups).
(iv) Q-polynomial maximal subgroups of a finite group.
(v) P-and Q-polynomial maximal subgroups of a finite group (= finite primitive P-and Q-polynomial Gelfand pairs).

The Problem (i) was studied by O'Nan-Scott, by Aschbacher and also by many others. (These are general theory of classifications of maximal subgroups of finite simple (or almost simple) groups, using the classification of finite simple groups.

Now, let us consider the case of the classical Lie type simple groups.

Liebeck: On the orders of maximal subgroups of finite classical groups (Proc. London Math. Soc.,1985).
Let G be a group with $G_0 \leq G \leq Aut(G_0)$ where G_0 is a classical type Lie type simple group. Let H be a maximal subgroup of G. Then we have either
(1) The pair (G, H) is in the known list,
or
(2) $|H|^3 \leq |G|$.

This result is obtained by using Aschbacher (1984) which depends on the classification of finite simple groups. The list for (1) is described explicitly for the special case $G = G_0 = PSp(2m, q)$ in the paper of Liebeck, but not for all other cases. I believe to obtain the complete listing for all cases may not be easy, but will not be completely impossible. We were able to considerably shorten the list in (1) assuming additionally that (G, H) is a Gelfand pair, i.e., H is a multiplicity-free subgroup of G.

For this purpose, the following paper is useful.

Liebeck-Pyber: Upper bounds for the number of conjugacy classes of a finite group (J. Algebra, 1997). Let G be a simple Lie type group of untwisted type of rank ℓ over the field of q elements, then the number of conjugacy classes of G is at most $(6q)^{\ell}$. (Some modifications for the twisted case.) If we assume that if (G, K) is a P-polynomial Gelfand pair, then, we can make the list of maximal subgroups of $G = G_0 = PSp(2m, q)$ by Liebeck(1985) shorter. (For other types, we can consult book of Kleidman-Liebeck (1990).)

If q is even,
(1) stabilizers of totally isotropic or non-singular subspaces of V,
(2) $(Sp(4m, q), Sp(2m, q) \wr S_2)$,
(3) $(Sp(2m, q), GL(m, q).2)$,
(4) $(Sp(2m, q), GU(m, q).2)$,
(5) $(Sp(4m, q), Sp(2m, q^2).2)$,
(6) $(Sp(2m, q^2), Sp(2m, q).2)$,
(7) $(Sp(2m, q), O^+(2m, q)), (Sp(2m, q), O^-(2m, q))$.

If q is odd,
(1) the stabilizers of totally isotropic or non-singular subspaces of V,
(2) $(PSp(4m,q), 2.PSp(2m,q) \wr S_2)$,
(3) $(PSp(2m,q), GL(m,q).2/Z)$ where $Z = \{\pm I\}$,
(4) $(PSp(2m,q), GU(m,q).2/Z)$ where $Z = \{\pm I\}$,
(5) $(PSp(2m,q^2), PSp(2m,q).2)$.

(Details need to be worked our more carefully, but this certainly works, at least if both q and ℓ are large.) The above is the short list of such possible maximal subgroups of $G = G_0 = PSp(2m,q)$ where multiplicity-free maximal subgroups are in this list. (Some of them are actually not multiplicity-free.)

The case of fixing totally isotropic subspace was studied very completely. Exactly speaking, we can know when they are multiplicity-free, as well as when they are P- or Q-polynomial, etc..)

As it was mentioned already, the case of fixing non-singular spaces is more delicate.

We note that the subfield case: $(Sp(2m,q^2), Sp(2m,q))$ was studied by Kawanaka: On subfield symmetric spaces over a finite field (Osaka J. Math.,1990). They are generally not multiplicity-free (at least if both q and m are not small). For other classical groups, it seems that it was studied by Lusztig (Representation Theory, 2000).
On the other hand, for the case $(Sp(4m,q), Sp(2m,q^2)$, it is actually shown by Lei Zhang (J. Algebra, 2013) that this is indeed multiplicity-free. It is very likely judging from various consideration that they are generally not P-polynomial nor Q-polynomial. (Basically all the irreducible representations of $Sp(2m,q)$ are known essentially by Deligne and Lusztig (although it is not so easy to use that result). I believe it is essentially possible to determine when each pair of the above list (of maximal subgroup of $G = PSp(2m,q)$) becomes Gelfand pair, or P-(and/or) Q-polynomial Gelfand pair or not should be possible. (But I have to say that I have not yet succeeded.) When we consider the P-polynomial property, we can use some more additional properties, such as

the unimodality of the subdegrees k_i. For the P-and Q-polynomial case, we can use the additional properties that some weaker unimodality for the m_i holds and all the irreducible representations appearing in the permutation character are rational. So, I believe it should be possible, at least in principle, to determine primitive P-and Q-polynomial Gelfand pairs. Then the ambitious thinking is to try to proceed to (i) non-primitive case, and proceed to (ii) multivariate P-and Q-polynomial case.

<u>Conclusions.</u> As for compact symmetric spaces vs. finite Gelfand pairs, roughly speaking, compact symmetric spaces are almost equivalent to general compact Gelfand pairs, and they seem always to have a similar property of multivariate P-and Q-polynomial property. So, we want to study finite Gefand pairs (or commutative association schemes) from the viewpoint of multivariate P-and Q-polynomial Gelfand pairs. I think it is important to try to understand the character tables of (known) finite simple groups from the viewpoint of multivariate P-and Q-polynomial association schemes. As the first step, how about for the character table of, say (the group association scheme of) $PSL(2, q)$ or other small controlling commutative association schemes?

Thank you very much

Appendix. Summary of the paper of Pantoja, Soto-Andrade, and Vargas [PSV] (2015).

Let q be odd, and let

$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and $Sp(2n, q^2) = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \, {}^t XJX = -J \right\}$.

Let $\mathcal{L}_{2n,q^2}$ be the set of all the totally isotropic subspaces of $V = F_{q^2}^{2n}$ with respect to this symplectic form. Then $Sp(2n, q^2)$ has the subgroup $Sp_0(2n, q)$ such that $Sp_0(2n, q) = \left\{ \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in Sp(2n, q^2) \right\}$. (This group is isomorphic to $Sp(2n, q)$ as an abstract group.) Then $\mathcal{L}_{2n,q^2}$ is decomposed into the orbits $\mathcal{O}_0, \mathcal{O}_1, ..., \mathcal{O}_n$ by the action of $Sp_0(2n, q)$. The action of $Sp_0(2n, q)$ on the orbit $\mathcal{O}_n$ s isomorphic to $Sp(2n, q)/GU(n, q)$. Note that [PSV] call $\mathcal{L}_{2n,q^2}$ the finite version of Siegel upper half space, and we regard the action of $Sp_0(2n, q)$ on $\mathcal{O}_n$ the finite version of Siegel upper half space.

Note that $h_0 = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$ be the hermitian form on $V = F_{q^2}^{2n}$. Then $\mathcal{O}_i$ is the set of elements of $\mathcal{L}_{2n,q^2}$ such that the restriction of h_0 is of rank i. Also, note that $Sp_0(2n, q)$ is the the intersection of the unitary group w.r.t. h_0 and $Sp(2n, q^2)$.

Possible finite version.

The following is a list of possible finite version of compact symmetric spaces. Here we assume $q = odd$.

$SU(n)/SO(n)$

$GL(n, q^2)/GL(n.q), GL(n, q^2)/GU(n, q), GU(n, q^2)/GL(n, q)$, $GU(n, q^2)/GU(n, q)$. (They are multiplicity-free.) Also, $GL(n, q)/O^*(n, q)$ (3 kinds), $GU(n, q)/O^*(n, q)$ (3 kinds), $Sp(2n, q^2)/Sp(2n, q)$, $O^*(n, q^2)/O^*(n, q)$ (3 kinds) could be candidates. (It seems they are not multiplicity-free in general.)

$SU(2n)/Sp(n)$

$GL(2n, q)/Sp(2n, q), GU(2n, q)/Sp(2n, q)$. (They are multiplicity-free.)

$SO(2n)/U(n)$

$GL(2n,q)/GL(n,q^2)$. (Multiplicity-free.) Also, $GU(2n,q)/GL(n,q^2)$ is a candidate. It looks $GU(2n,q)/GU(n,q^2), GU(2n,q)/GU(n,q^2)$ do not appear in the list. (Am I correct? Yes.) $Sp(4n,q)/Sp(2n,q^2), O^*(2n,q)/O^*(n,q^2)$ (3 kinds), $O^*(2n,q)/GU(n,q)$ (3 kinds), could also be candidates. (I expect some cases are likely to be multiplicity-free. Some cases are indeed so, but need to check more carefully.) I am really hoping that the case $O^*(2n,q)/GU(n,q)$ (3 kinds) become actually multiplivity-free.

$Sp(n)/U(n)$

$Sp(2n,q)/GU(n,q)$. (There will be some overlaps with the case of $SO(2n)/U(n)$.)

There are many works try to study finite version of compact symmetric spaces. First, let me mention two of them, for example.

Finite symmetric spaces. (Cf. G. Lusztig: Symmetric spaces over a finite field (1990).) Let G be a (connected) reductive group over a finite field F_q (q odd) with a given involution $\theta : G \to G$ defined over F_q. The pair (G, θ) will be called a symmetric space (over F_q); we shall fix a closed subgroup K of the fixed point set G^θ such that K is defined over F_q and K contains the identity component $(G^\theta)^0$ of G^θ. (So, the symmetric space is essentially the homogeneous space G/K.)

Quandles. (Cf. D. Joyce: Simple quandles, J. Alg. (1982).) Let G be a finite group and let ϕ be an automorphism of G. If K is a subgroup of the fixed subgroup G^ϕ of G, Then the homogeneous quandle $Q = G/K$ is determined by the quandle triple (G, K, ϕ).

Although these definitions are reasonable, I noticed that many important examples of Gelfand pairs are missing in this context. (Note that compact symmetric spaces are classified and they are all Gelfand pairs.) We could avoid this situation, if we consider the homogeneous space by the normalizers of any subgroup. This is not the main point I want to discuss today, but I believe we should consider general Gelfand pairs in the finite case, hoping to classify them all at least for the primitive case.

Now, we want to discuss the classification problem of finite Gelfand pairs (G, K), in particular, of finite primitive Gelfand pairs.

The classification of maximal subgroups, in particular for finite (almost) simple groups has been developed extensively using the classification of finite simple groups. Notably, the O'Nan-Scott theorem and the work of Aschbacher: On the maximal subgroups of the finite classsical groups (1984). (These results are very involved and very difficult. Here, we briefly review this situation.