Counting edges of different types in a local graph of a Grassmann graph

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Outline

Let V denote a finite-dimensional vector space over a finite field. The corresponding projective geometry P is the poset consisting of the subspaces of V, with partial order by inclusion.

A Grassmann graph Γ associated with P is known to be distance-regular.

Pick distinct vertices x, y of Γ that are not adjacent and at distance less than the diameter of Γ. Consider a two-vertex stabilizer $\text{Stab}(x, y)$ in $GL(V)$. It is known that there are five orbits of the $\text{Stab}(x, y)$-action on the local graph of x, denoted by $\Gamma(x)$.

Outline

In this talk we define three types of edges in $\Gamma(x)$, namely type 0, type $+$, type $-$. For adjacent vertices $w, z \in \Gamma(x)$ such that w, z are equidistant from y, the type of the edge wz depends on the subspaces $w + z, w, z, w \cap z$ and their intersections with y.

For each pair of orbits $\mathcal{O}, \mathcal{N}$ in $\Gamma(x)$, and a given vertex $w \in \mathcal{O}$, we find the number of vertices $z \in \mathcal{N}$ adjacent to w such that the edge wz has (i) type 0, (ii) type $+$, (iii) type -. This is the main result of the talk.

To do this, we make heavy use of a subalgebra $\mathcal{H}$ of $\mathrm{Mat}_P(\mathbb{C})$ that contains some matrices that are closely related to the five orbits in $\Gamma(x)$.

Preliminaries

Let $\Gamma = (X, E)$ denote a graph with no loops or multiple edges, vertex set X, edge set E, and path-length distance function ∂.

By the *diameter* D of Γ, we mean the value $\max\{\partial(x, y) \mid x, y \in X\}$.

For a vertex $x \in X$, let the set $\Gamma(x)$ consist of vertices in X that are adjacent to x. By the *local graph of* x, we mean the subgraph of Γ induced on $\Gamma(x)$.

Distance-regular graph

Definition

We say that Γ is *distance-regular* whenever for all $0 \leqslant h, i, j \leqslant D$ and all $x, y \in X$ such that $\partial(x, y) = h$, the cardinality of the set

$$\{z \in X \mid \partial(x, z) = i,\ \partial(y, z) = j\}$$

depends only on h, i, j. This cardinality is denoted by $p^h_{i,j}$.

From now on, we assume that Γ is distance-regular.

Projective geometry P

This talk is about a class of distance-regular graphs called the Grassmann graphs. These graphs are defined from projective geometries, which we describe below.

Let $\mathbb{F}_q$ denote a finite field with q elements. Let $n \geqslant 1$. Let V denote a vector space of dimension n over $\mathbb{F}_q$.

Let the set $P = P_q(n)$ consist of the subspaces of V. The set P, together with inclusion partial order, is a poset called a *projective geometry*.

For $u, v \in P$, we say that v *covers* u whenever $u \subseteq v$ and $\dim v = \dim u + 1$.

The Grassmann graph $J_q(n, k)$

For $0 \leqslant \ell \leqslant n$, let the set P_ℓ consist of elements of P that have dimension ℓ. We have a partition

$$P = \bigcup_{\ell=0}^{n} P_\ell.$$

We now define the Grassmann graph $J_q(n, k)$.

Definition

For $n > k \geqslant 1$, the Grassmann graph $J_q(n, k)$ has vertex set $X = P_k$. Two vertices $x, y \in X$ are adjacent whenever $x \cap y \in P_{k-1}$.

The graph $J_q(n, k)$ is known to be distance-regular.

The Grassmann graph $J_q(n, k)$

The graph $J_q(n, k)$ is isomorphic to $J_q(n, n-k)$. Without loss, we assume that $n \geqslant 2k$.

The case $n = 2k$ is somewhat special, so for the rest of the paper, we assume that $n > 2k$.

Under this assumption, the diameter of $J_q(n, k)$ is k.

From now on, we assume that $\Gamma = J_q(n, k)$ with $k \geqslant 3$.

Symmetry of P

Recall that $GL(V)$ consists of invertible $\mathbb{F}_q$-linear maps from V to V.

The action of $GL(V)$ on V induces a permutation action of $GL(V)$ on P.

This action preserves dimension and the inclusion partial order.

The action of $GL(V)$ on X preserves the path-length distance ∂.

The stabilizer Stab(x, y) and its action on $\Gamma(x)$

Definition

For distinct $x, y \in X$ let $\mathrm{Stab}(x, y)$ denote the subgroup of $GL(V)$ consisting of the elements that fix both x and y. We call $\mathrm{Stab}(x, y)$ the *stabilizer of x and y*.

Pick distinct $x, y \in X$. Our next goal is to describe the orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$.

The cases $\partial(x, y) = 1$ and $\partial(x, y) = k$ are somewhat special. For the rest of the talk, we assume that $1 < \partial(x, y) < k$.

Stab(x, y)-action on $\Gamma(x)$

Definition

For $x, y \in X$ such that $1 < \partial(x, y) < k$, define

$$\mathcal{B}_{xy} = \{z \in \Gamma(x) \mid \partial(z, y) = \partial(x, y) + 1\},$$
$$\mathcal{C}_{xy} = \{z \in \Gamma(x) \mid \partial(z, y) = \partial(x, y) - 1\},$$
$$\mathcal{A}_{xy} = \{z \in \Gamma(x) \mid \partial(z, y) = \partial(x, y)\}.$$

Lemma (S. 2024)

For $x, y \in X$ such that $1 < \partial(x, y) < k$, the sets $\mathcal{B}_{xy}, \mathcal{C}_{xy}$ are orbits of the Stab(x, y) action on $\Gamma(x)$.

As we will see, the set $\mathcal{A}_{xy}$ is the disjoint union of three orbits. To describe these orbits, we refine the partition of P described earlier.

Partition of P

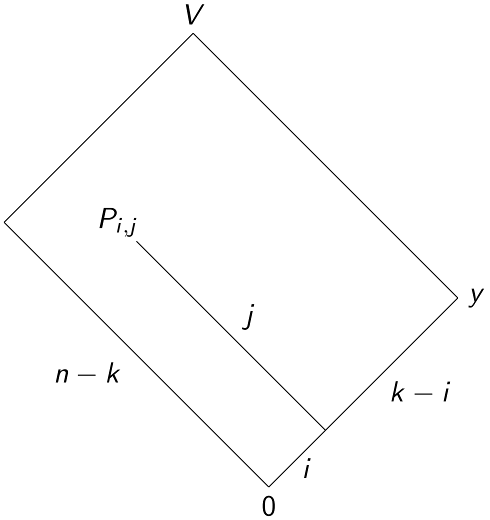
Recall the partition

$$P = \bigcup_{\ell=0}^{n} P_\ell.$$

We now refine this partition. For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant n-k$, define

$$P_{i,j} = \{u \in P \mid \dim(u \cap y) = i,\ \dim u = i + j\}.$$

Partitions of P



$/$-cover and $\backslash$-cover

Earlier, we described the covering relation on P. We now give a refinement of this covering relation.

Lemma (S. 2025+)

Let $u, v \in P$ such that v covers u. Write

$$u \in P_{i,j}, \qquad v \in P_{r,s}.$$

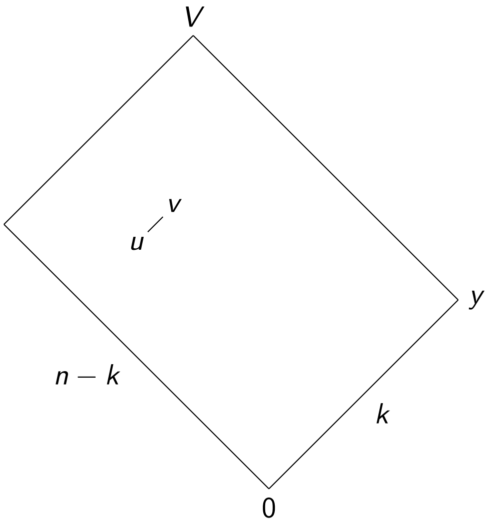
Then either (i) $r = i + 1$ and $s = j$, or (ii) $r = i$ and $s = j + 1$.

Definition

We say that v *$/$-covers u* whenever (i) holds, and v *$\backslash$-covers u* whenever (ii) holds.

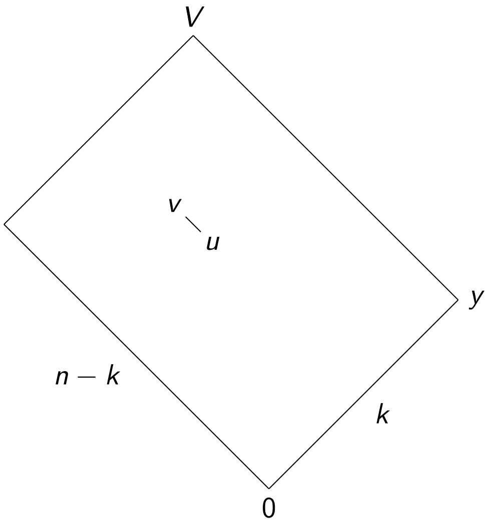
/-cover and \-cover

v /-covers u:



/-cover and \-cover

v \-covers u:



Stab(x, y)-action on $\Gamma(x)$

Definition

For $x, y \in X$ such that $1 < \partial(x, y) < k$, define

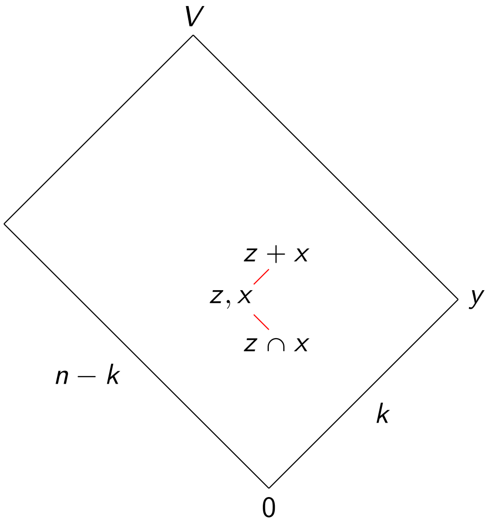
$$\mathcal{A}^0_{xy} = \{z \in \Gamma(x) \mid z + x \text{ /-covers each of } z \text{ and } x\text{, which } \backslash\text{-covers } z \cap x\},$$
$$\mathcal{A}^+_{xy} = \{z \in \Gamma(x) \mid z + x \ \backslash\text{-covers each of } z \text{ and } x\},$$
$$\mathcal{A}^-_{xy} = \{z \in \Gamma(x) \mid \text{each of } z \text{ and } x \text{ /-covers } z \cap x\}.$$

Lemma (S. 2025+)

For $x, y \in X$ such that $1 < \partial(x, y) < k$, the set $\mathcal{A}_{xy}$ is the disjoint union of the sets $\mathcal{A}^0_{xy}, \mathcal{A}^+_{xy}, \mathcal{A}^-_{xy}$.

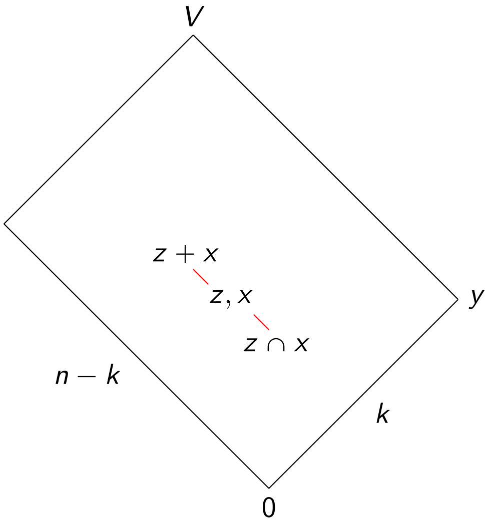
Stab(x, y)-action on $\Gamma(x)$

$z \in \mathcal{A}^0_{xy}$:



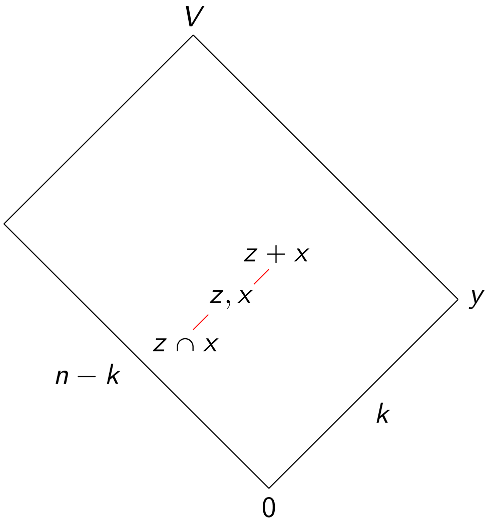
Stab(x, y)-action on $\Gamma(x)$

$z \in \mathcal{A}_{xy}^{+}$:



Stab(x, y)-action on $\Gamma(x)$

$z \in \mathcal{A}_{xy}^{-}$:



Stab(x, y)-action on $\Gamma(x)$

Lemma (S. 2025+)

For $x, y \in X$ such that $1 < \partial(x, y) < k$, the following sets are orbits of the $\mathrm{Stab}(x, y)$*-action on* $\Gamma(x)$*:*

$$\mathcal{B}_{xy}, \qquad \mathcal{C}_{xy}, \qquad \mathcal{A}^0_{xy}, \qquad \mathcal{A}^+_{xy}, \qquad \mathcal{A}^-_{xy}.$$

Furthermore, these orbits form an equitable partition of $\Gamma(x)$.

Structure constants

We present the corresponding structure constants. In what follows, we use the notation

$$[m] = \frac{q^m - 1}{q - 1} \qquad\qquad (m \in \mathbb{Z}).$$

Referring to the table below, for each orbit $\mathcal{O}$ in the header column, and each orbit $\mathcal{N}$ in the header row, the $(\mathcal{O}, \mathcal{N})$-entry gives the number of vertices in $\mathcal{N}$ that are adjacent to a given vertex in $\mathcal{O}$. Write $i = \partial(x, y)$.

Structure constants

	$\mathcal{B}_{xy}$	$\mathcal{C}_{xy}$	$\mathcal{A}^0_{xy}$	$\mathcal{A}^+_{xy}$	$\mathcal{A}^-_{xy}$
$\mathcal{B}_{xy}$	$q^{i+1}[k-i]$ $+q^{i+1}[n-k-i]-q-1$	0	0	$q[i]$	$q[i]$
$\mathcal{C}_{xy}$	0	$2q[i-1]$	$2q^i-q-1$	$q^{i+1}[n-k-i]$	$q^{i+1}[k-i]$
$\mathcal{A}^0_{xy}$	0	$2[i]-1$	$2q^i-q-2$	$q^{i+1}[n-k-i]$	$q^{i+1}[k-i]$
$\mathcal{A}^+_{xy}$	$q^{i+1}[k-i]$	$[i]$	$(q-1)[i]$	$q[n-k]-q-1$	0
$\mathcal{A}^-_{xy}$	$q^{i+1}[n-k-i]$	$[i]$	$(q-1)[i]$	0	$q[k]-q-1$

Types of edges in $\Gamma(x)$

We now define three types of edges in $\Gamma(x)$; as we will see, these types are related to the orbits $\mathcal{A}^0_{xy}, \mathcal{A}^+_{xy}, \mathcal{A}^-_{xy}$.

Definition

Pick $x, y \in X$ such that $1 < \partial(x, y) < k$. For adjacent $w, z \in \Gamma(x)$ such that $\partial(w, y) = \partial(z, y)$, we say that the edge wz has

- (i) type 0 whenever $w + z$ /-covers each of w and z, which \-covers $w \cap z$,
- (ii) type $+$ whenever $w + z$ \-covers each of w and z,
- (iii) type $-$ whenever each of w, z /-covers $z \cap x$.

For adjacent $w, z \in \Gamma(x)$ such that $\partial(w, y) = \partial(z, y)$, the edge wz has exactly one of the three types above.

Types of edges in $\Gamma(x)$

Assume that $1 < \partial(w, y) = \partial(z, y) < k$. The edge wz has

(i) type 0 if and only if $w \in \mathcal{A}^0_{zy}$,

(ii) type $+$ if and only if $w \in \mathcal{A}^+_{zy}$,

(iii) type $-$ if and only if $w \in \mathcal{A}^-_{zy}$.

Pick a pair of orbits $\mathcal{O}, \mathcal{N}$. Pick $w \in \mathcal{O}$. Our next goal is to find the number of vertices $z \in \mathcal{N}$ adjacent to w such that the edge wz has (i) type 0, (ii) type $+$, (iii) type $-$.

To do this, we make heavy use of the subalgebra $\mathcal{H}$ of $\mathrm{Mat}_P(\mathbb{C})$.

Let $\mathrm{Mat}_P(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices with rows and columns indexed by P and all entries in $\mathbb{C}$.

Throughout the talk we use the following notation. For $u, v \in P$ and $M \in \mathrm{Mat}_P(\mathbb{C})$, let $M_{u,v}$ denote the (u, v)-entry of M.

Definition

We define diagonal matrices $K_1, K_2 \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u \in P$ their (u, u)-entries are

$$\left(K_1\right)_{u,u} = q^{\frac{k}{2}-i}, \qquad \left(K_2\right)_{u,u} = q^{j-\frac{n-k}{2}},$$

where $u \in P_{i,j}$. Note that K_1, K_2 are invertible.

The algebra $\mathcal{H}$

Definition

We define matrices $L_1, L_2, R_1, R_2 \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u, v \in P$ their (u, v)-entries are

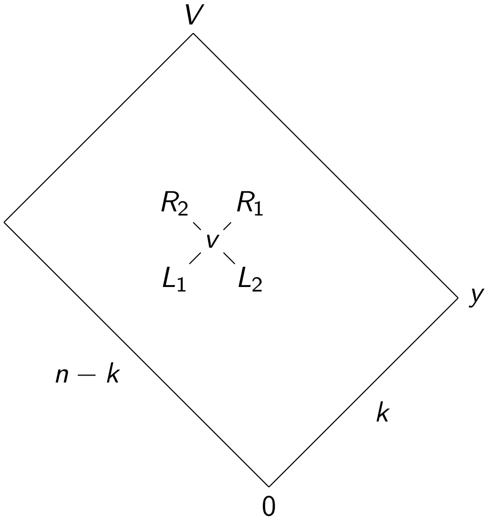
$$(L_1)_{u,v} = \begin{cases} 1 & \text{if } v \text{ /-covers } u, \\ 0 & \text{if } v \text{ does not /-cover } u, \end{cases}$$

$$(L_2)_{u,v} = \begin{cases} 1 & \text{if } v \text{ \\-covers } u, \\ 0 & \text{if } v \text{ does not \\-cover } u, \end{cases}$$

$$(R_1)_{u,v} = \begin{cases} 1 & \text{if } u \text{ /-covers } v, \\ 0 & \text{if } u \text{ does not /-cover } v, \end{cases}$$

$$(R_2)_{u,v} = \begin{cases} 1 & \text{if } u \text{ \\-covers } v, \\ 0 & \text{if } u \text{ does not \\-cover } v. \end{cases}$$

The algebra $\mathcal{H}$



The algebra $\mathcal{H}$

Definition

Let $\mathcal{H}$ denote the subalgebra of $\mathrm{Mat}_P(\mathbb{C})$ generated by $L_1, L_2, R_1, R_2, K_1^{\pm 1}, K_2^{\pm 1}$.

Next we define matrices R, L, F^0, F^+, F^-.

Definition

Define

$$R = L_1 R_2, \qquad L = R_1 L_2.$$

Note that $R, L \in \mathcal{H}$.

The algebra $\mathcal{H}$

Definition

Define matrices $F^0, F^+, F^- \in \mathrm{Mat}_P(\mathbb{C})$ as follows. For $u, v \in P$ their (u, v)-entries are

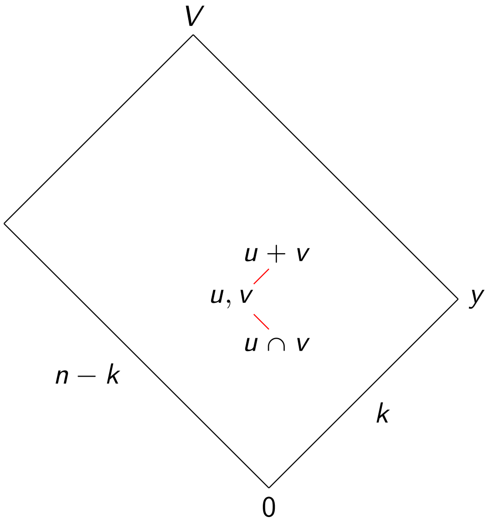
$$(F^0)_{u,v} = \begin{cases} 1 & \text{if } u+v \ /\text{-covers each of } u \text{ and } v, \text{ which } \backslash\text{-covers } u \cap v, \\ 0 & \text{otherwise,} \end{cases}$$

$$(F^+)_{u,v} = \begin{cases} 1 & \text{if } u+v \ \backslash\text{-covers each of } u \text{ and } v, \\ 0 & \text{otherwise,} \end{cases}$$

$$(F^-)_{u,v} = \begin{cases} 1 & \text{if each of } u \text{ and } v \ /\text{-covers } u \cap v, \\ 0 & \text{otherwise.} \end{cases}$$

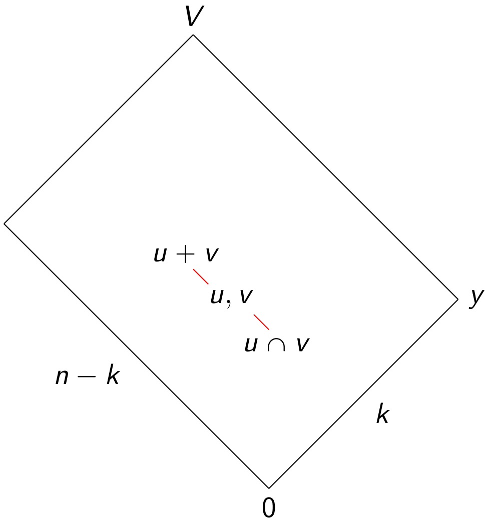
The algebra $\mathcal{H}$

$\left(F^0\right)_{u,v} = 1$:



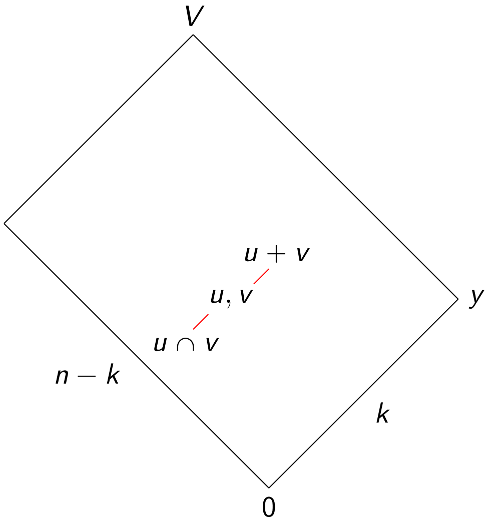
The algebra $\mathcal{H}$

$\left(F^{+}\right)_{u,v} = 1$:



The algebra $\mathcal{H}$

$\left(F^{-}\right)_{u,v} = 1$:



The algebra $\mathcal{H}$

Lemma (S. 2025+)

The matrices F^0, F^+, F^- are contained in $\mathcal{H}$.

Next we express F^0, F^+, F^- in terms of the generators of $\mathcal{H}$.

Lemma (S. 2025+)

The matrices F^0, F^+, F^- satisfy

$$\begin{aligned} F^0 &= L_1R_1 - R_1L_1 + (q-1)^{-1}\left(q^{\frac{n}{2}}K_1^{-1}K_2 - q^{\frac{k}{2}}K_1 - q^{\frac{n-k}{2}}K_2 + I\right) \\ &= R_2L_2 - L_2R_2 + (q-1)^{-1}\left(q^{\frac{n}{2}}K_1K_2^{-1} - q^{\frac{k}{2}}K_1 - q^{\frac{n-k}{2}}K_2 + I\right), \\ F^+ &= L_2R_2 - q^{\frac{k}{2}}(q-1)^{-1}K_1\left(q^{\frac{n-k}{2}}K_2^{-1} - I\right), \\ F^- &= R_1L_1 - q^{\frac{n-k}{2}}(q-1)^{-1}\left(q^{\frac{k}{2}}K_1^{-1} - I\right)K_2. \end{aligned}$$

The algebra $\mathcal{H}$

Next we recall the generating set for the center of $\mathcal{H}$.

Lemma (S. 2025+)

The center of $\mathcal{H}$ is generated by the following three elements:

$$\Omega_0 = q^{-\frac{n}{2}}\left((q-1)F^0K_1^{-1}K_2^{-1} + q^{\frac{n-k}{2}}K_1^{-1} + q^{\frac{k}{2}}K_2^{-1} - K_1^{-1}K_2^{-1}\right),$$

$$\Omega_1 = q^{-\frac{n-k}{2}}\Bigg(qF^0K_2^{-1} + (q-1)F^-K_2^{-1} + (q-1)^{-1}\left(q^{\frac{k}{2}+1}K_1K_2^{-1} + q^{\frac{n}{2}+1}K_1^{-1} - qK_2^{-1}\right)\Bigg) - q(q-1)^{-1}I,$$

$$\Omega_2 = q^{-\frac{k}{2}}\Bigg(qF^0K_1^{-1} + (q-1)F^+K_1^{-1} + (q-1)^{-1}\left(q^{\frac{n-k}{2}+1}K_1^{-1}K_2 + q^{\frac{n}{2}+1}K_2^{-1} - qK_1^{-1}\right)\Bigg) - q(q-1)^{-1}I.$$

The algebra $\mathcal{H}$

Lemma (S. 2025+)

We have

$$F^0 = \frac{q^{\frac{n}{2}}\Omega_0 K_1 K_2 - q^{\frac{k}{2}} K_1 - q^{\frac{n-k}{2}} K_2 + I}{q-1},$$

$$F^+ = \frac{q^{\frac{k}{2}}\Omega_2 - (q-1)^{-1}\left(q^{\frac{n}{2}+1}\left(\Omega_0 K_2 + K_2^{-1}\right) - 2q^{\frac{k}{2}+1} I\right)}{q-1} K_1,$$

$$F^- = \frac{q^{\frac{n-k}{2}}\Omega_1 - (q-1)^{-1}\left(q^{\frac{n}{2}+1}\left(\Omega_0 K_1 + K_1^{-1}\right) - 2q^{\frac{n-k}{2}+1} I\right)}{q-1} K_2.$$

The matrices F^0, F^+, F^- mutually commute.

Relations that involve R, L, F^0, F^+, F^-

Next we present some useful relations that involve R, L, F^0, F^+, F^-.

Theorem (S. 2025+)

We have

$$q^2 RF^0 - F^0 R + \left(q^{\frac{k}{2}} K_1 + q^{\frac{n-k}{2}} K_2 - (q+1) I\right) R = 0;$$

$$qRF^+ - F^+ R - F^0 R + (q-1)^{-1} \left(q^{\frac{n}{2}+1} K_2^{-1} K_1 - q^{\frac{k}{2}} K_1 - q^{\frac{n-k}{2}} K_2 + I\right) R = 0;$$

$$qRF^- - F^- R - F^0 R + (q-1)^{-1} \left(q^{\frac{n}{2}+1} K_1^{-1} K_2 - q^{\frac{k}{2}} K_1 - q^{\frac{n-k}{2}} K_2 + I\right) R = 0;$$

$$q^2 F^0 L - LF^0 + \left(q^{\frac{k}{2}+1} K_1 + q^{\frac{n-k}{2}+1} K_2 - (q+1) I\right) L = 0;$$

$$qF^+ L - LF^+ - LF^0 + (q-1)^{-1} \left(q^{\frac{n}{2}+1} K_2^{-1} K_1 - q^{\frac{k}{2}+1} K_1 - q^{\frac{n-k}{2}+1} K_2 + I\right) L = 0;$$

$$qF^- L - LF^- - LF^0 + (q-1)^{-1} \left(q^{\frac{n}{2}+1} K_2^{-1} K_1 - q^{\frac{k}{2}+1} K_1 - q^{\frac{n-k}{2}+1} K_2 + I\right) L = 0.$$

Relations that involve R, L, F^0, F^+, F^-

Theorem (S. 2025+)

We have

$$0 = LR - qF^+F^- - (q-1)^{-1}\Big(\big(q^{\frac{n}{2}+1}K_1^{-1}K_2 - q^{\frac{n-k}{2}+1}K_2\big)F^+ + \big(q^{\frac{n}{2}+1}K_1K_2^{-1} - q^{\frac{k}{2}+1}K_1\big)F^-\Big) - (q-1)^{-2}\Big(q^{\frac{n}{2}+1}K_1K_2 - q^{n-\frac{k}{2}+1}K_1 - q^{\frac{n+k}{2}+1}K_2 + q^{n+1}I\Big);$$

$$0 = qRL - \big(F^0\big)^2 - F^0F^+ - F^0F^- - F^+F^- - (q-1)^{-1}\Big(\big(q^{\frac{k}{2}}K_1 + q^{\frac{n-k}{2}}K_2 - 2I\big)F^0 + \big(q^{\frac{k}{2}}K_1 - I\big)F^+ + \big(q^{\frac{n-k}{2}} - I\big)F^-\Big) - (q-1)^{-2}\big(q^{\frac{n}{2}}K_1K_2 - q^{\frac{k}{2}}K_1 - q^{\frac{n-k}{2}}K_2 + I\big).$$

Entries of matrices that involve F^0, F^+, F^-

For $w \in \mathcal{A}_{xy}$, we find the (w, x)-entries of many matrices that involve F^0, F^+, F^-.

Theorem (S. 2025+)

Referring to the table below, for each matrix M in the header column and each orbit $\mathcal{O}$ in the header row, the $(M, \mathcal{O})$-entry gives the (w, x)-entry of M when $w \in \mathcal{O}$. Write $i = \partial(x, y)$.

Entries of matrices that involve F^0, F^+, F^-

	$\mathcal{A}^0_{xy}$	$\mathcal{A}^+_{xy}$	$\mathcal{A}^-_{xy}$
$(F^0)^2$	$2q^i - q - 2$	0	0
F^0F^+	0	$(q-1)[i]$	0
F^0F^-	0	0	$(q-1)[i]$
F^+F^0	0	$(q-1)[i]$	0
$(F^+)^2$	$q^{i+1}[n-k-i]$	$q[n-k] - q^i - q$	0
F^+F^-	0	0	0
F^-F^0	0	0	$(q-1)[i]$
F^-F^+	0	0	0
$(F^-)^2$	$q^{i+1}[k-i]$	0	$q[k] - q^i - q$

Types of edges in $\Gamma(x)$

Theorem (S. 2025+)

Referring to the table below, for each orbit $\mathcal{O}$ in the header column, and each orbit $\mathcal{N}$ in the header row, the $(\mathcal{O}, \mathcal{N})$-entry gives a 3×1-matrix that satisfies the following: each entry of the matrix corresponds to the number of vertices in $\mathcal{N}$ that share an edge of type $0, +, -$ respectively with a given vertex in $\mathcal{O}$. We omit the entries $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Write $i = \partial(x, y)$.

Types of edges in $\Gamma(x)$

	$\mathcal{B}_{xy}$	$\mathcal{C}_{xy}$	$\mathcal{A}^0_{xy}$	$\mathcal{A}^+_{xy}$	$\mathcal{A}^-_{xy}$
$\mathcal{B}_{xy}$	$\begin{pmatrix} 2q^{i+1}-q-1 \\ q^{i+2}[n-k-i-1] \\ q^{i+2}[k-i-1] \end{pmatrix}$				
$\mathcal{C}_{xy}$		$\begin{pmatrix} 0 \\ q[i-1] \\ q[i-1] \end{pmatrix}$			
$\mathcal{A}^0_{xy}$			$\begin{pmatrix} 2q^i-q-2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ q^{i+1}[n-k-i] \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ q^{i+1}[k-i] \end{pmatrix}$
$\mathcal{A}^+_{xy}$			$\begin{pmatrix} 0 \\ (q-1)[i] \\ 0 \end{pmatrix}$	$\begin{pmatrix} (q-1)[i] \\ q[n-k]-q^i-q \\ 0 \end{pmatrix}$	
$\mathcal{A}^-_{xy}$			$\begin{pmatrix} 0 \\ 0 \\ (q-1)[i] \end{pmatrix}$		$\begin{pmatrix} (q-1)[i] \\ 0 \\ q[k]-q^i-q \end{pmatrix}$

Summary

We recalled the projective geometry $P_q(n)$ and the Grassmann graph $J_q(n,k)$.

For vertices x, y such that $1 < \partial(x,y) < k$, we recalled the $\text{Stab}(x,y)$-action on $\Gamma(x)$ and its five orbits.

We defined three types of edges in $\Gamma(x)$. For each pair of orbits $\mathcal{O}, \mathcal{N}$ and a given vertex $w \in \mathcal{O}$, we found the number of vertices $z \in \mathcal{N}$ adjacent to w such that the edge wz has (i) type 0, (ii) type $+$, (iii) type $-$.

To do this, we brought in the subalgebra $\mathcal{H}$ of $\text{Mat}_P(\mathbb{C})$; the algebra $\mathcal{H}$ contains useful matrices like R, L, F^0, F^+, F^-. We presented many relations that involve R, L, F^0, F^+, F^-. For $w \in \mathcal{A}_{xy}$, we found the (w,x)-entries of many matrices that involve F^0, F^+, F^-.

Thank you for your attention.